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THE SUMMER MEETING AT WILLIAMSTOWN

The fortieth summer meeting and seventeenth colloquium of the Society was held at Williams College, Williamstown, Massachusetts, from Tuesday to Friday, September 4–7, 1934. The meeting was preceded by that of the Mathematical Association of America. About two hundred seventy persons attended the meetings, among whom were the following one hundred sixty-nine members of the Society:

C. R. Adams, R. B. Adams, R. P. Agnew, E. B. Allen, D. B. Ames, R. C. Archibald, R. P. Bailey, D. H. Ballou, R. A. Beaver, A. A. Bennett, Theodore Bennett, William Beverley, Harry Birchenough, Garrett Birkhoff, G. D. Birkhoff, L. M. Blumenthal, Joseph Bowden, J. W. Bower, H. W. Brinkmann, B. H. Brown, H. S. Brown, R. E. Bruce, J. A. Bullard, L. H. Bunyan, W. H. Bussey, S. S. Cairns, W. D. Cairns, R. H. Cameron, B. H. Camp, C. C. Camp, G. A. Campbell, F. E. Carr, W. B. Carver, Abraham Cohen, L. W. Cohen, H. R. Cooley, L. P. Copeland, Byron Cosby, M. J. Cox, H. B. Curry, D. R. Curtiss, J. H. Curtiss, H. H. Dalaker, D. R. Davis, F. F. Decker, L. L. Dines, H. A. DoBell, H. L. Dorwart, Jesse Douglas, Arnold Dresden, L. A. Dye, H. T. Engstrom, G. W. Evans, H. P. Evans, H. S. Everett, O. J. Farrell, D. A. Flanders, L. R. Ford, W. B. Ford, R. M. Foster, C. W. Franklin, Philip Franklin, T. C. Fry, A. S. Galbraith, C. A. Garabedian, H. M. Gehman, J. J. Gergen, B. P. Gill, D. C. Gillespie, J. G. Hardy, Alan Hazeltine, O. C. Hazlett, E. R. Hedrick, M. R. Hestenes, T. R. Hollcroft, E. V. Huntington, W. A. Hurwitz, J. I. Hutchinson, M. H. Ingraham, R. L. Jeffery, B. W. Jones, L. S. Kennison, B. F. Kimball, J. R. Kline, H. L. Krall, J. H. Kusner, W. D. Lambert, A. E. Landry, R. E. Langer, V. S. Lawrence, A. M. M. Lehr, Harry Levy, Hans Lewy, E. R. Lorch, N. H. McCoy, W. H. McEwen, L. A. MacColl, Saunders MacLane, Morris Marden, A. E. Meder, G. M. Merriman, H. J. Miles, E. C. Molina, C. N. Moore, Richard Morris, Marston Morse, David Moskovitz, E. J. Moulton, J. R. Musselman, D. S. Nathan, J. H. Neelley, E. P. Northrop, C. O. Oakley, Oystein Ore, F. W. Owens, H. B. Owens, F. W. Perkins, H. B. Phillips, Hillel Poritsky, G. B. Price, Tibor Radó, S. M. Rambo, O. J. Ramler, J. F. Randolph, W. R. Ransom, H. W. Raudenbush, M. S. Rees, W. T. Reid, C. E. Rhodes, R. G. D. Richardson, D. E. Richmond, Robin Robinson, W. E. Roth, H. G. Russell, S. A. Schelkunoff, C. E. Seely, I. M. Sheffer, J. A. Shohat, L. L. Silverman, W. G. Simon, T. M. Simpson, M. M. Slotnick, A. W. Smith, C. E. Smith, W. M. Smith, Virgil Snyder, E. S. Sokolnikoff, I. S. Sokolnikoff, A. H. Sprague, E. C. Stokes, M. M. Sullivan, T. Y. Thomas, J. I. Tracey, F. E. Ulrich, J. L. Walsh, L. E. Ward, J. F. Wardwell, A. C. Washburne, C. W. Watkeys, M. S. Webster, V. H. Wells, H. S. White, Hassler Whitney, Norbert Wiener, A. H. Wilson, E. W. Wilson, R. G. Wood, E. R. Worthington, C. H. Yeaton.

The colloquium lectures, on *Fourier transforms in the complex domain*, as originally planned by Professor Norbert Wiener and Dr. R. E. A. C. Paley (now deceased) were delivered on Tuesday afternoon and Wednesday, Thursday, and Friday mornings. These lectures, which were attended by eighty-nine members of the Society, will appear in book form as volume 19 of the Colloquium Publications of this Society.

By invitation of the Committee on Program, Professor J. A. Shohat gave on Thursday afternoon an address entitled *On the expansion of functions in series of orthogonal polynomials*.

Wednesday afternoon was devoted to an excursion over the Taconic Trail to the new Bennington College for Women and Bennington battlefield. An interesting account of the battle of Bennington was given by Mr. W. C. Hart, a lineal descendant of one of the leaders in this engagement. On Wednesday evening Dr. and Mrs. Charles Louis Safford gave a very fine organ and song recital in Chapin Hall; Dr. Safford is the organist of Williams College.

On Thursday evening a joint dinner of the Society and the Mathematical Association of America was held at the Greylock Hotel. Professor E. V. Huntington acted as toastmaster and called upon President Tyler Dennett of Williams College, on Professor E. R. Hedrick representing the Society, and on President Arnold Dresden of the Association. Professor Virgil Snyder presented a resolution expressing appreciation of the cordial hospitality of the College. He called attention to the fact that this was the second time that the Society had held its summer meeting at Williamstown, the first occasion having been in 1905. Seven members present had attended the earlier meeting and recalled it with pleasure.

At the close of the colloquium on Thursday morning a group photograph of those in attendance was taken. The American Mathematical Monthly expects to reproduce this photograph.

The dormitories of Williams College were open to members and their guests. A delightful social center was furnished through the generosity of the Governors of the Faculty Club who extended its privileges to the members of the group. Here the ladies of the Department of Mathematics of Williams College served tea every afternoon. The college tennis courts were avail-

able free of charge and the Taconic Golf Club offered reduced rates to members of the Society.

The Council met on the evening of September 4th and various actions taken at this time were reported to the Society.

The following twelve persons were elected to membership in the Society:

Mr. Carliton Eugene Buell, Washington University;
Mr. John Hamilton Curtiss, Harvard University;
Professor Emmet Roach Elliott, Hampden-Sidney College;
Mr. Henry Gerhardt, Mobile, Alabama;
Mr. John Pawl Gill, Jr., University of Alabama;
Mr. Paul Matthew Hummel, Ohio State University;
Dr. Louis Kattsoff, University of Virginia;
Dr. Ellis R. Ott, University of Buffalo;
Mr. Billy James Pettis, Brown University;
Mr. Walter Stevens Turpin, Johns Hopkins University;
Dr. George Cuthbert Webber, University of Chicago;

As nominee on the Sustaining Membership of the Western and Southern Life Insurance Company: Mr. Richard James Learson, Jr.

It was reported that in accordance with a reciprocity agreement, the following person has joined the Society;

Professor C. Wilhelm Oseen, Nobelinstitutet, Stockholm, Sweden.

The Secretary announced the election of the following persons to membership by mail vote of the Council since its meeting in April:

Dr. John Henry Augustus Brahtz, U.S. Bureau of Reclamation, Denver, Colorado;
Professor Raymond Donald Douglass, Massachusetts Institute of Technology;
Dr. Kurt Gödel, Princeton University;
Professor Harriette Grace Holt, University of Wisconsin Extension;
Professor O. M. Lind, Ch'an Cheng-Lob' Philosophical Centre, Thaton, Burma;
Mr. Vicente Mills, Department of Agriculture and Commerce, Manila, Philippine Islands;
Dr. Daniel T. Sigley, University of Kansas City;
Professor John H. Van Vleck, University of Wisconsin;
Dr. Stefan Emanuel Warschawski, Columbia University;
Professor Earl Francis Watts, Massachusetts Institute of Technology.

The following appointments by President Coble were reported; as representative at the inauguration of W. O. Menden-

hall as President of Whittier College, September 21, 1934, Professor E. R. Hedrick; as additional member of the Committee on the Award of the Cole Prize in Algebra, Professor R. W. Barnard.

On Thursday morning there was a business meeting of the Society. At this meeting on recommendation of the Council, the by-laws were changed so as to assign $5 of the dues of each member to the Bulletin.

The titles and cross references to abstracts of the papers read at the sessions follow below. The papers were read as follows: papers numbered 1 to 6 in the Section for Analysis on Tuesday afternoon, Professor G. D. Birkhoff presiding; papers numbered 7 to 10 in the Section for Algebra on Tuesday afternoon, Professor Oystein Ore presiding; papers numbered 11 to 17 in the Section for Analysis on Wednesday morning, Professor Marston Morse presiding; papers 18 to 20 in the Section for Algebra and Number Theory on Wednesday morning, Professor H. W. Brinkmann presiding; papers 21 to 23 on Thursday morning, Professor Virgil Snyder presiding; papers 24 to 37 on Thursday afternoon, Professor Tibor Radó presiding; papers 38 to 66 on Friday morning, Professor Arnold Dresden presiding. Mr. Curtiss was introduced by Professor J. L. Walsh, Mr. Elliott by Professor E. R. C. Miles, Mr. Seale by Professor J. V. Uspensky, Mr. Reingold by Professor I. A. Barnett, Mr. Friedman by Professor B. P. Gill, Mr. Levinson by Professor Norbert Wiener, Mr. Van Schaack by Professor Marston Morse, Dr. Terry by Professor H. R. Brahana. Mr. Trump's paper (number 9) was communicated by Professor M. H. Ingraham. Papers whose abstract numbers are followed by *t* were read by title.

1. *Interpolation in regularly distributed points*, by Mr. John Curtiss. (Abstract No. 40-9-263.)

2. *On approximation by polynomials to a function analytic in a simply connected region*, by Professor O. J. Farrell. (Abstract No. 40-9-265.)

3. *Analytic approximations to manifolds*, by Dr. Hassler Whitney. (Abstract No. 40-9-293.)

4. *On the zeros of the derivative of a rational function*, by Professor Morris Marden. (Abstract No. 40-9-283.)

5. *A mixed boundary-value problem for the heat equation*, by Mr. E. R. Elliott. (Abstract No. 40-9-264-*t*.)

6. *Integration of functions with values in a Banach space*, by Mr. Garrett Birkhoff. (Abstract No. 40-9-299.)

7. *Metabelian groups and pencils of bilinear forms*, by Professor H. R. Brahana. (Abstract No. 40-9-261-*t*.)

8. *The non-singular case of the equivalence of pairs of Hermitian matrices*, by Professor M. H. Ingraham. (Abstract No. 40-7-242.)

9. *On a reduction of matrices by the group of matrices commutative with a given matrix*, by Mr. P. L. Trump. (Abstract No. 40-7-251.)

10. *A simple proof of Minkowski's theorem concerning non-homogeneous linear forms*, by Mr. R. Q. Seale. (Abstract No. 40-9-287-*t*.)

11. *Linear differential equations with almost periodic coefficients*, by Dr. R. H. Cameron (National Research Fellow). (Abstract No. 40-5-217.)

12. *Seminvariants of a system of linear homogeneous differential equations of the second order*, by Mr. Haim Reingold and Professor I. A. Barnett. (Abstract No. 40-7-249.)

13. *A priori limitations for solutions of elliptic Monge-Ampère equations*, by Dr. Hans Lewy. (Abstract No. 40-9-278.)

14. *A singular solution of a functional equation*, by Mr. L. B. Robinson. (Abstract No. 40-9-286-*t*.)

15. *An inclusion theorem in the theory of summable series*, by Dr. H. L. Garabedian. (Abstract No. 40-7-245-*t*.)

16. *On the nature of the "reflection" in a plane boundary corresponding to the boundary condition* $\partial u/\partial n + \text{const.}\, u = 0$, by Dr. Hillel Poritsky. (Abstract No. 40-9-305.)

17. *The theory of the second variation for the non-parametric problem of Bolza*, by Dr. W. T. Reid. (Abstract No. 40-9-306.)

18. *On certain rings and differential ideals*, by Professor N. H. McCoy. (Abstract No. 40-9-279.)

19. *Zeroless solutions of systems of linear equations in a finite field*, by Professor B. P. Gill. (Abstract No. 40-9-272.)

20. *Sufficient conditions for the solvability of some Pellian equations*, by Mr. Bernard Friedman. (Abstract No. 40-9-270.)

21. *The inter-deducibility of the new Hilbert-Bernays theory*

and Principia Mathematica, by Professor E. V. Huntington. (Abstract No. 40-7-246.)

22. *Foundations of the theory of abstract sets from the standpoint of combinatory logic*, by Professor H. B. Curry. (Abstract No. 40-9-262.)

23. *On the asymptotic solutions of ordinary differential equations, with reference to the Stokes phenomenon about a singular point*, by Professor R. E. Langer. (Abstract No. 40-9-275.)

24. *Note on linear operations on functions of bounded variation*, by Professor J. J. Gergen. (Abstract No. 40-9-271.)

25. *A local solution of the difference equation* $\Delta y = f$ *and of related equations*, by Professor I. M. Sheffer. (Abstract No. 40-9-288.)

26. *The problem of Bolza in the calculus of variations in parametric form*, by Dr. M. R. Hestenes (National Research Fellow). (Abstract No. 40-9-273.)

27. *A note on permutable self-adjoint transformations in Hilbert space*, by Dr. E. R. Lorch (National Research Fellow). (Abstract No. 40-7-248.)

28. *On the vanishing of a function over an interval*, by Mr. Norman Levinson. (Abstract No. 40-9-276.)

29. *On properties of regions which persist in the subregions bounded by level curves of the Green's function*, by Professor L. R. Ford. (Abstract No. 40-9-267-*t*.)

30. *An operational solution of the Maxwell field equations*, by Dr. E. P. Northrop. (Abstract No. 40-9-285.)

31. *Sufficient conditions in the problem of Lagrange without assumptions of normalcy*, by Professor Marston Morse. (Abstract No. 40-9-303-*t*.)

32. *Convergence properties of Fourier series*, by Professor Otto Szász. (Abstract No. 40-9-297-*t*.)

33. *A formula for finding the skewness of the distribution made up of a finite number of samples*, by Professor W. D. Baten. (Abstract No. 40-9-298-*t*.)

34. *The fundamental groups of group manifolds*, by Professor P. A. Smith. (Abstract No. 40-9-296-*t*.)

35. *The critical point theory under general boundary conditions*, by Professor Marston Morse and Mr. G. B. Van Schaack. (Abstract No. 40-9-302-*t*.)

36. *The boundary conditions in the critical point theory*, by

Professor Marston Morse and Mr. G. B. Van Schaack. (Abstract No. 40-9-301-*t*.)

37. *Short proof of Kellogg's lemma*, by Professor G. C. Evans. (Abstract No. 40-9-304-*t*.)

38. *On nets of algebraic surfaces*, by Professor T. R. Hollcroft. (Abstract No. 40-9-274.)

39. *A six-color problem*, by Professor Philip Franklin. (Abstract No. 40-9-269.)

40. *On certain types of continuous transformations of metric spaces*, by Professor W. A. Wilson. (Abstract No. 40-7-252-*t*.)

41. *Some unique separation theorems for graphs*, by Dr. Saunders MacLane. (Abstract No. 40-9-280.)

42. *On the expansion of harmonic functions in series of harmonic polynomials belonging to a simply connected region*, by Professor O. J. Farrell. (Abstract No. 40-9-266-*t*.)

43. *Dynamics of multi-phase commutator motors* (preliminary report), by Mr. W. H. Ingram. (Abstract No. 40-7-247-*t*.)

44. *A theorem on the magnitude of Fourier transforms*, by Mr. Norman Levinson. (Abstract No. 40-9-277-*t*.)

45. *On the partial derivatives of harmonic functions as functionals of the determining boundaries*, by Dr. Mildred M. Sullivan (National Research Fellow). (Abstract No. 40-9-289-*t*.)

46. *On cyclic fields*, by Professor A. A. Albert. (Abstract No. 40-9-254-*t*.)

47. *A greatest common divisor process for integers of a quadratic field*, by Professor A. A. Albert. (Abstract No. 40-9-255-*t*.)

48. *The principal matrices of a Riemann matrix*, by Professor A. A. Albert. (Abstract No. 40-9-256-*t*.)

49. *On the power series for elliptic functions*, by Professor E. T. Bell. (Abstract No. 40-7-244-*t*.)

50. *Concerning the order of a linear transformation with coefficients in a* $GF(p)$, by Professor H. R. Brahana. (Abstract No. 40-9-260-*t*.)

51. *A classification of finite distributive lattices*, by Dr. Saunders MacLane. (Abstract No. 40-9-281-*t*.)

52. *Groups in which the squares of the elements are dihedral groups*, by Professor G. A. Miller. (Abstract No. 40-9-284-*t*.)

53. *Abelian subgroups of the I-group of the abelian group of order* p^n *and type* $1, 1, \cdots$, by Dr. Henrietta Terry. (Abstract No. 40-9-290-*t*.)

54. *On the expansion of the remainder in the Newton-Cotes formula*, by Professor J. V. Uspensky. (Abstract No. 40-9-291-*t*.)

55. *Irreducibility of polynomials of degree n which assume the same value n times*, by Professor Louis Weisner. (Abstract No. 40-9-292-*t*.)

56. *On the abstract properties of linear dependence*, by Dr. Hassler Whitney. (Abstract No. 40-9-294-*t*.)

57. *Multiply connected sets*, by Dr. R. E. Basye. (Abstract No. 40-9-257-*t*.)

58. *Transfinite extensions of the Jordan-Hölder theorem*, by Mr. Garrett Birkhoff. (Abstract No. 40-5-218-*t*.)

59. *An application of a theorem of K. Gödel*, by Dr. L. M. Blumenthal (National Research Fellow). (Abstract No. 40-9-258-*t*.)

60. *A decomposition of the spherical surface $S_{2,r}$ by means of tetrahedral quadruples*, by Dr. L. M. Blumenthal (National Research Fellow). (Abstract No. 40-9-259-*t*.)

61. *On Minkowski's definition of length and area*, by Professor Philip Franklin. (Abstract No. 40-9-268-*t*.)

62. *A projective generalization of metrically defined associate surfaces*, by Professor M. L. MacQueen. (Abstract No. 40-9-282-*t*.)

63. *Prime ends and indecomposability*, by Dr. N. E. Rutt. (Abstract No. 40-7-243-*t*.)

64. *The analysis of certain curves by means of derived local separating points*, by Dr. G. E. Schweigert. (Abstract No. 40-7-250-*t*.)

65. *On the imbedding of metric sets in euclidean space*, by Professor W. A. Wilson. (Abstract No. 40-7-253-*t*.)

66. *The equation of Izumi and a contour problem*, by Mr. L. B. Robinson. (Abstract No. 40-9-295-*t*.)

J. R. KLINE,
Associate Secretary

STONE ON HILBERT SPACE

Linear Transformations in Hilbert Space and their Applications to Analysis. By M. H. Stone. New York, American Mathematical Society, 1932. viii +622 pp.

The attitude of mathematicians towards the applications varies. Some feel a need of rationalizing their personal interests by pointing out their importance for the applications, and the applications know them not. Others assert with pride that their results cannot possibly be applied, and fate whimsically harnesses their purest dreams to the chariot of industry. Whether we want our results to be applied or not, we probably all agree that the applications have a way of posing stimulating questions which spur the progress of our science. Thus we probably owe more of our advance in analysis to the prying curiosity of the physicist than to any other agent.

We recall how the problem of wave motion and heat conduction led to the development of the function concept and the introduction of orthogonal series, basic elements of present day analysis. Dirichlet's problem in potential theory had a profound influence on the calculus of variations and led to Fredholm's theory of integral equations. We know what happened to this theory in the masterly hands of Hilbert, how it became a theory of orthogonal transformations and reduction of quadratic forms, how Hilbert revived the interest in orthogonal functions, and created the atmosphere which stimulated, for example, F. Riesz in his basic discoveries of different types of convergence in function spaces. One is perhaps justified in regarding F. Riesz's *Systèmes Linéaires d'une Infinité d'Inconnues* as the climax of the development of this period, though its novel points of view really mark it off as the forerunner of much of the modern theory.

It is only fair to admit that much of this development seemed of little interest to the average physicist who, of course, knew that Dirichlet's problem could be solved long before we actually solved it. But there was heavy weather ahead for the physicists; the foundations of physics were taken for a ride, and a bewildered physicist had to reorient himself in a strange world, perhaps not created by a mathematician, but at least one where the mathematician played the role of the little tin god on wheels. Space and time became differential geometry and tensor analysis. Energy and matter disappeared in a cloud of operators, abstract algebra, probabilities, and boundary value problems.

The analytical problems of quantum and wave mechanics can be thought of in terms of linear transformations in a space of infinitely many dimensions with certain specified properties, the abstract Hilbert space. To Hilbert and his pupils we owe a theory of completely continuous and of bounded transformations of this space. An excellent account of this theory is to be found in the Encyclopädie article of Hellinger and Toeplitz. The modern theory of unbounded transformations is connected chiefly with the names of T. Carleman, J. von Neumann, F. Riesz, M. H. Stone, and A. Wintner. Carleman's work on integral equations started in 1916; a powerful analytical technique enabled him to push this theory far beyond the bounds reached by the Hilbert

school. His older work is collected in his *Équations Intégrales Singulières* [Upsala, 1923]; his later results have been presented in outline in a Poincaré Institute lecture and in congress addresses. If Carleman be regarded as the heir of the Fredholm tradition, von Neumann's leaning towards algebraic methods and abstract formulations makes him a worthy successor of Hilbert. His investigations in this field, started in 1927, culminated in his book *Mathematische Grundlagen der Quantenmechanik* (Springer, 1932). Wintner also leans towards the Hilbert tradition as formulated by Toeplitz; his investigations are collected in his book *Spektraltheorie der Unendlichen Matrizen* (Hirzel, 1929). F. Riesz, finally, has shown how the simple and elegant methods which he employed in the bounded case can be used also for unbounded transformations [Acta Szeged, vol. 5 (1930)].

The first results of Stone's investigations appeared in three notes in the Proceedings of the National Academy of Sciences in 1929–30, of which the third one contained the spectral resolution of unitary groups of transformations in Hilbert space. The present book is an outgrowth of these preliminary communications. The author has set himself the task of giving a thorough and systematic exposition of the main features of the theory, based on his own investigations, but also containing the principal results of Carleman and von Neumann. Let it be said at once that he has accomplished this task in an excellent manner. It is only to be regretted that considerations of space forced him to omit all group theoretical questions.

It is difficult for an outsider to ascribe credit where credit is due. A sizeable portion of the material in Stone's book is of course introductory or common property. Other portions are avowedly based upon outside sources, but the material is so well assimilated, digested, and developed that complete unity is preserved. Finally the author's own investigations appear to be the main source for large portions of the book, especially in chapters five to seven and in the middle of chapter ten. The author has evidently tried to be strictly fair in his references to the work of other writers; in some places one feels, however, that the theorems have been improved beyond recognition. That the author is indebted in various ways to the writings of von Neumann is obvious, and he makes generous acknowledgment of the fact. It is also obvious that he has penetrated the difficult writings of Carleman, and has shouldered the important task of bringing Carleman's theory in line with the later development. This is particularly desirable since Carleman's work has not always received adequate recognition by the Hilbert school.

In what follows, we shall outline the contents of Stone's treatise.

The first chapter gives the definition of the abstract Hilbert space and its main properties, as well as various realizations of the space. This space $\mathfrak{H}$ is characterized by five properties: (i) it is a linear vector space, (ii) to any pair of elements f and g corresponds a complex-valued functional (f, g), with certain properties, which defines the metric in $\mathfrak{H}$, the norm of f being $\|f\| = (f, f)^{1/2}$, (iii) for every n there exist n linearly independent elements, (iv) $\mathfrak{H}$ is separable, and (v) complete.

Transformations are introduced in Chapter 2. A careful study is made of the notions of domain, range, extension, inverse, closure, continuity, linearity, and adjointness in connection with transformations. T_1 is adjoint to T_2, in symbols $T_1 \wedge T_2$, if $(T_1 f, g) = (f, T_2 g)$ in the domains of the transformations.

To every T corresponds a unique adjoint T^* such that any T_1 with $T_1 \wedge T$ admits of T^* as an extension. The author calls a transformation H symmetric [Hermitian in usual terminology] if $H \wedge H$ and the smallest linear manifold containing the domain of H is $\mathfrak{H}$ itself. A transformation H is maximal if it admits of no proper symmetric extension, essentially self-adjoint if $H^* = (H^*)^*$, and self-adjoint if $H = H^*$. The rest of the chapter deals with bounded transformations [$\|Tf\| < C\|f\|$], projections [bounded, maximal symmetric transformations E such that $E \cdot E = E$], isometric transformations [$(Uf, Ug) = (f, g)$], and unitary transformations [isometric with range and domain $\mathfrak{H}$]. An isometric non-unitary transformation has a deficiency-index (m, n), m and n being the dimension numbers of the manifolds orthogonal to the domain and range of Uf, respectively.

Chapter 3 illustrates these concepts by examples from the theory of infinite matrices, integral operators including Fourier and Hankel transforms, and differential operators.

In Chapter 4 the main problem is the study of the inverse of the transformation $T_l \equiv T - lI$, where l is a complex number. This leads to the spectrum [point, continuous, and residual spectra in general] and to the resolvent R_l of T. Particular reference is made to the symmetric case.

The detailed study of self-adjoint transformations is taken up in Chapter 5, based upon the earlier publications of the author. The gist of the method goes back to Stieltjes and Carleman, but notation and terminology agree essentially with those of von Neumann. To a self-adjoint H corresponds a family of projections $E(\lambda)$, known as a resolution of the identity such that

$$(R_l f, g) = \int_{-\infty}^{\infty} (\lambda - l)^{-1} d(E(\lambda)f, g).$$

The domain of H consists of those and only of those elements of $\mathfrak{H}$ for which $\int_{-\infty}^{\infty} \lambda^2 d\|E(\lambda)f\|^2 < \infty$, and for such f's and arbitrary g's

$$(Hf, g) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)f, g).$$

The points of discontinuity of $E(\lambda)$ furnish the point spectrum, while points of continuity not interior to intervals of constancy give the continuous spectrum. There is no residual spectrum.

Chapter 6 deals with operational calculus. The basic problem is to give a sense to the notion of a function of a transformation. Let H be self-adjoint, $E(\lambda)$ its resolution of the identity, and $F(\lambda)$ a complex-valued function of a real variable; then we define $F(H)$ by

$$(F(H)f, g) = \int_{-\infty}^{\infty} F(\lambda) d(E(\lambda)f, g),$$

where the integral is of the Radon-Stieltjes type, and the domain of the transformation $F(H)$ has to be determined.

On the basis of this operational calculus the author solves in Chapter 7 the difficult problem of unitary equivalence of two self-adjoint transformations H_1 and H_2, that is, the question of the existence of a unitary transformation U such that $H_1 = UH_2U^{-1}$. For the bounded case this problem was solved by

Hellinger and Hahn. The solution leads to a complete characterization of the various possible types of self-adjoint transformations.

In Chapter 8, the basic idea is that of permutability of two transformations which leads to various complications in the unbounded case. It contains further a study of unitary and normal $[T \cdot T^* = T^* \cdot T]$ transformations, the theory of which is closely allied to the self-adjoint case.

The general symmetric transformations are taken up in Chapter 9. With a symmetric H the author, following von Neumann, associates an isometric transformation V such that $H \equiv i(I+V)(I-V)^{-1}$, and assigns to H the same deficiency-index as V has. This gives a basis for the classification. It is followed by a theory of formally real transformations, and an abstract formulation of the approximation theorems of Carleman with additions and elaborations.

Chapter 10 is devoted to applications. It occupies 218 pages, more than one third of the whole book, and still it is not long enough. The author starts with the theory of unbounded integral operators, that is, essentially the theory of integral equations with kernels of the Carleman type, which is a special case and also a forerunner of the general theory. This is followed by the theory of differential operators in which the author gives a very thorough and systematic discussion of differential systems of the first and second order. These operators provide fairly simple examples of symmetric transformations with deficiency-index (1, 1) or (2, 2). The author has here carried investigations initiated by Weyl and Carleman to an elegant finish. The last topic treated is the theory of Jacobi matrices which includes the moment problem for infinite intervals and the theory of continued fractions. Here we encounter the results of Carleman, Hamburger, R. Nevanlinna, and M. Riesz in new clothing. There are many other applications one should have liked to have included here. The important applications to the ergodic hypothesis [the long wished-for child which had so many fathers!] and to non-linear differential equations in general, obviously could not be included since the basic papers appeared while the book was in press.

To sum up, Professor Stone has done us a real service by providing a clear, systematic, and readable account of this important theory which is still in statu nascendi. The book is not always easy reading, but the author is fair to the reader: nothing essential is withheld, the terminology is clearly defined and strictly adhered to, there are enough facts provided for the imagination to feed on, but no loose statements.* We hope that the author will give us a second volume with further applications and aspects of this fascinating theory before long.

EINAR HILLE

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* According to the author there is at least one mistake in the book: on p. 229, line 4 from the bottom, the reference should read "Theorem 4.18" instead of "Theorem 2.18." The reviewer has not found any errors.

CAJORI'S EDITION OF NEWTON'S PRINCIPIA

Sir Isaac Newton's Mathematical Principles of Natural Philosophy. Translated into English by Andrew Motte in 1729. The translations revised, and supplied with an historical and explanatory appendix. By Florian Cajori. University of California Press, 1934. xxxvi+680 pp.

The *magnum opus* of Newton, the *Principia*, is too well known to the scientific world to justify any elaborate description in a review of this nature. Suffice it to recall the fact that the first edition appeared in 1687 and that the errors and imperfections were numerous. Out of 494 pages W. W. Rouse Ball found that 397 were modified or corrected in the second edition—the one due to Cotes, of whom Newton remarked "If Cotes had lived, we had known something." After the second edition appeared Newton sent to Cotes a list of twenty errata, to which the latter added an equal number and remarked, "I made some Hundreds, with which I never acquainted You." Newton's only comment seems to have been, "Its impossible to print a book without some faults." All this goes to show that even the gods can err, and that a new edition of a classic of 1687 was justified nearly two and a half centuries later. It may also be remarked that Bentley, Master of Trinity, suggested that Cotes should include in the second edition "An alphabetical Index." Had Professor Cajori lived to see his own edition through the press, he might have recalled Bentley's good advice, especially as concerns the appendix. He died, however, before his manuscript was sent to the printer and, as the editor of the present revision, Professor R. T. Crawford, tells us, he left only the manuscript of the notes which form the appendix, even a preface being lacking.

This edition is a revision of Motte's translation from the third (Latin) edition (1726), use being made of Thorp's later translation (ed. 2, 1802). The only changes made in the text are such as relate to the symbolism and the phraseology, these being limited to such as are necessary to express the original ideas in modern form.

The work consists of the English text (626 pp.) and Professor Cajori's appendix (54 pp.). In the front matter are included Halley's "Ode Dedicated to Newton," translated into English by Professor Leon J. Richardson, and the prefaces to the first three editions, the one to the second edition, by Cotes (pp. xx–xxxiii), being of major importance. It is concerning the appendix, which contains the new material, however, that the reader will wish for information. A few of the most important items will therefore be mentioned, the space allowed for this purpose being limited.

In this appendix Professor Cajori discusses (p. 629 seq.) at length and with thoroughness the preface which Cotes wrote for the second edition, and in particular the theory of vortices which Descartes had published in 1644, more than twenty years before the *Principia* appeared. This theory had soon thereafter been made known in England by such scholars as Henry More (Cambridge) and Joseph Glanvill (Oxford) and by the publication (London, 1682) of Bonet's Latin translation of Rohault's French textbook on physics. Professor Cajori calls attention to the popular nature of this theory as compared with

Newton's laws of inverse squares in gravitational attraction. In spite of the difficulty experienced by the average student in reading the original text, however, the work had the support of such leaders as Halley, Whiston, Cotes, Taylor, Saunderson, Robert Smith, and both David and James Gregory. Nevertheless, even Whiston asserted that the Cartesian theory should first be taught "to the youth of the university" because it was easier, and Rohault's textbook continued in use, even in Newton's scientific home, until forty-three years after the *Principia* first appeared, and three years after its celebrated author found his last resting place in Westminster Abbey.

Professor Cajori gives a brief but clear summary (pp. 632–635) of the opinions of Newton's contemporaries and immediate successors concerning his real position respecting gravity. As a matter of fact, Newton himself confessed in a letter to Bentley, in January 1693 (N.S.), that he did not pretend to know its cause and, as is well known, he was opposed to hypotheses. All this is familiar to students of the Newtonian philosophy, but the facts are here well documented and an interesting comparison with the views of Einstein is made. The nature of gravitation is also discussed (p. 636) with respect to action at a distance—a theory due rather to Cotes in his preface to the second edition—and the relation of this to Maxwell's electromagnetic theory of light.

Although it is impossible in the space allowed to discuss at any length the most important topics of the appendix, the following items may at least be briefly mentioned as being of major importance, the numbers being those of the respective paragraphs:

11. Definition I, "The quantity of matter is the measure of the same, arising from its density and bulk conjointly," with reference to modern sources.

13. Absolute time, absolute space, place, and absolute motion, naturally leading to a statement as to the recent work of Ernest Mach, Michelson, Morley, Fitzgerald, Lodge, Lorentz, de Sitter, and Einstein.

14. Laws of motion, with reference to Kaufmann, Weyl, Eddington, and Einstein.

22. Kepler's problem, "To find the place of a body moving in a given ellipse at any assigned time," together with the scholium, essentially the solution of $x-e \sin x=z$, where e and z are known. A brief history of the problem is given.

23. "To find the motion of the apsides in orbits approaching very near to circles," with references to the study of S. B. Gaythorpe (1925), to Newton's work on the moon problem, and to the unfortunate controversy with Flamsteed. A valuable bibliography is included.

30. Fixed infinitesimals (Book II, Lemma II). Fluxions in the *Principia*, the relations between Newton and Leibniz. A summary of the history of the Newtonian calculus and a brief discussion of the obscurity of Newton. The treatment is much abridged, doubtless because of the amount of literature available.

35. On the resistance in a rare medium of a globe and a cylinder (Book II, proposition 34), with Newton's remark, "This proposition I conceive may be of use in the building of ships." This discussion is one of Professor Cajori's chief contributions. In it he considers the nature of surfaces of least resistance. His historical notes include the results set forth by Bolza.

38–49. In these notes Professor Cajori has entered one of the fields of his greatest interest—the history of physical problems. Those here discussed concern the velocity of sound, a field in which Newton did not meet with much success; the Huygenian telescope, the great length of the instrument being rendered unnecessary by Dolland's invention of the achromatic lens; the earth-moon test of the law of gravitation; the figure of the earth, and the problem of three bodies.

Newton's idea of God, his attitude toward hypotheses (*Hypotheses non fingo*), and his views concerning causality are considered in the closing pages (668–680). These naturally lead to a study of the workings of Newton's mind in general rather than to the mathematics of the *Principia*, but the discussion is none the less interesting, psychologically and historically.

Looking at the book from the mechanical point of view of the bibliophile, it is one of the finest pieces of printing and binding to be found among the mathematical books that have come from any press in this country. Congratulations are due to the University of California Press for such an excellent product. The lack of an index is its chief defect.

Thanks and congratulations are also due to the editor, Professor Crawford, for the painstaking care with which he has performed a very difficult task, and for his contribution to such a monumental work.

As to the work of Professor Cajori, he has here and more especially in his treatise on the history of mathematical symbols built for himself a noble and enduring monument, showing himself the leader among the historians of mathematics in this country and a conscientious and thorough scholar.

DAVID EUGENE SMITH

TARDI ON GEODESY

Traité de Géodésie. By P. Tardi. Paris, Gauthier-Villars, 1934. xxx+732 pp. 150 frs.

In a recent issue (vol. 40, No. 9, p. 644) of this Bulletin the reviewer considered Hopfner's book on geodesy and called attention to its highly theoretical nature. Captain Tardi's book is of a much more practical character. A good deal of space is devoted to directions for the use of instruments in the field. In this respect it resembles standard American text books on geodesy. This reviewer, however, will deal chiefly with the mathematical and theoretical portions.

Chapter 3, *Rappel de quelques théories mathématiques*, goes into some rather elementary matters of logarithms, trigonometry, infinite series, approximate formulas, the theory of errors, and the method of least squares. Many of the formulas are given without proof. The reviewer does not like the treatment of the Gaussian law of frequency of error, but this is a subject about which notoriously tastes differ.

Chapter 8 deals with the geometrical properties of the ellipsoid of revolution and of lines on its surface and applies the results to the calculation of a geodetic triangulation. Proofs are mostly given, at least in outline, but Gauss's fundamental theorem regarding the substitution of a sphere of the same mean curvature as the surface itself for the surface in question is merely stated with-

out a precise reference to Gauss's memoir, or to any other place where the proof may be found. The theorem applies, of course, only to infinitesimal line elements but in practice its application is extended to lines of small but finite length; the errors committed by extending the theorem to lines of finite length are stated without proof but the source of the little table of errors is given. Legendre's theorem for the calculation of a spherical triangle like a plane one is developed and pushed to a higher approximation than is given by the usual practice of subtracting one-third of the excess from each spherical angle. The formula for the transfer of geographic coordinates (given the latitude and longitude of a first point and the distance and azimuth of a second point from the first to find the latitude and longitude of the second point) is developed in a simple form and one that seems to be sufficiently accurate in practice, although the reviewer has not had a chance to test it. The solution of this problem has been put in many forms, each aiming at a combination of theoretical rigor with simplicity in numerical computation; almost every national geodetic organization has its own variation of the process. The form given by Captain Tardi and reached by simple mathematical means seems to be due to him. It appears to be convenient and surprisingly accurate, considering the small amount of algebraic manipulation needed to obtain it.

Chapter 9 deals with the plane representation (mapping) of the terrestrial ellipsoid. However, it is mapping in a very restricted sense. Rectangular cartesian coordinates are much more convenient than latitude and longitude, that is, for the average person, to whom meridians and parallels are very imaginary lines indeed. During the World War much use was made of maps on which a rectangular grid was superposed, points being specified by their rectangular coordinates in the grid. Similar systems for the special benefit of land surveyors have been used in Germany for more than a century. Each of the formerly separate German states has its own origin and axes. The errors in surveying arising from identifying the point on the map, conceived enlarged to approximately natural scale, with the corresponding point on the ground can be made small, if the area considered is small. The practice in surveying varies according to the length of line and the accuracy desired. Short lines may sometimes be treated as if they were exactly identical with the line between corresponding points on the enlarged map. Longer lines or lines for which some accuracy is needed require certain corrections. It is desirable to choose such a system of mapping as shall make these corrections as small and as easy to compute as possible. Captain Tardi gives a good account of the subject, an account that has already been found useful by the U.S. Coast and Geodetic Survey, which has devised a set of mapping or coordinate systems for the benefit of the surveyors of the various states. Two states have officially adopted the systems proposed for them and others will doubtless follow, for much work has been done using the various proposed systems more or less unofficially. This fact explains why this particular chapter is treated at disproportionate length in this review. In using the chapter some misprints have been detected but they will probably cause little difficulty.

Chapters 10 and 11 on the calculation of triangulation and its adjustments by least squares are about what would be found in a good American textbook.

Chapter 16 on gravity is not a particularly satisfactory one. There is a brief

account of the Newtonian potential and of the earth's field of force. Some of the simpler formulas are derived, but other formulas are stated without proof. Among these are numbers (396) and (398), which must surely be wrong, for the terms added together are not of the same physical dimensions throughout; also number (399), in which a sign is wrong.

Chapter 17 gives some technical details on gravity and sketches the theory of the Eötvös torsion balance.

Chapter 18 sketches methods for determining the figure of the earth by triangulation and astronomical methods and gives some of the results. Chapter 19 returns to gravity and shows how the flattening of the earth may be determined from gravity observations. No mention is made of Stokes's formula whereby the irregularities of the geoid may be determined by gravity observations. Doubtless, however, the author was justified in omitting it; the actual realization is not for the immediate future, in spite of the fact that gravity can now be determined at sea; moreover, the mathematics involved is rather beyond the scope of the work.

The author has already considered a geophysical question, namely isostasy, in connection with the figure of the earth, whether determined by triangulation or by gravimetric methods. Chapter 20 treats another geophysical question, the rigidity of the earth. The determinations of this rigidity by means of the long-period ocean tides, the tidal deflections of the vertical and the variation of latitude are outlined. Naturally a subject like this, which bristles with difficulties, cannot be treated exhaustively in a few pages. The reader, however, is at least introduced to the subject and made to realize that geodesy is a branch of geophysics. When the geodesist undertakes to interpret his data he inevitably becomes a geophysicist.

Perhaps this review, by discussing only the mathematical side, the side on which the book is probably the weakest, may have given a too unfavorable impression of the work. It is an excellent book on the whole, somewhat more comprehensive and more advanced than the average American textbook. A few more bibliographical references would improve it. It ought by means of successive revisions to have a long life, though probably in the nature of the case not as long as the book that it was written to supplant, namely, Francoeur's *Geodesy*, the first edition of which appeared in 1835. Successive revisions of Francoeur underwent comparatively slight changes; finally the stock of the latest revised edition became exhausted and question arose whether to revise Francoeur still further or to start afresh; the decision was to start afresh and Captain Tardi, who had had considerable practical experience and training in theory was selected for the work. Changes are coming so fast nowadays that, however excellent Tardi's book may be, it is hardly likely to attain the astonishing longevity of Francoeur's.

W. D. LAMBERT

SHORTER NOTICES

Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, Elliptischen, und anderen Funktionen. By Keiichi Hayashi. Berlin, Springer, 1933. 66 pp.

This book contains ten major tables and several minor ones. The first and most extensive of the major tables bears the title *Tafeln für die Differenzenrechnung.*

By starting with the Taylor expansion of $f(x \pm nh)$ and using the definition of differences, it is an easy matter to derive formulas for differences in terms of derivatives. The most general formula of this kind is

$$\Delta^m_{x \pm nh} = \frac{[\cdots]h^m}{m!} f^{(m)}(x) \pm \frac{[\cdots]h^{m+1}}{(m+1)!} f^{(m+1)}(x) + - \cdots$$

where the brackets represent coefficients which depend upon m and n. For example,

$$\Delta^5_{x-2h} = \frac{[120]}{5!} h^5 f^v(x) - \frac{[-360]}{6!} h^6 f^{vi}(x) + \frac{[1680]}{7!} h^7 f^{vii}(x) - + \cdots$$

$$= h^5 f^v(x) + \tfrac{1}{2} h^6 f^{vi}(x) + \tfrac{1}{3} h^7 f^{vii}(x) + \cdots .$$

The main table under the calculus of differences contains the numerical values of the bracket coefficients for $m=1$ to $m=12$ and $n=0$ to $n=50$. A second table gives the values of the coefficients

$$\frac{[\cdots]}{m!}, \quad \frac{[\cdots]}{(m+1)!}, \cdots ;$$

and a third table gives the values of

$$\frac{[\cdots]}{m!} h^m, \quad \frac{[\cdots]h^{m+1}}{(m+1)!}, \cdots$$

for $h=0.01$.

The second major table is entitled *Interpolationstafeln für* $J_0(x)$, $J_1(x)$. The main table gives the values of certain coefficients used in computing differences from the values of Bessel functions. Other tables give the values of $J_0(nh)$, $J_1(nh)$, $\cdots$, $J_6(nh)$ and of $J_n'(0)$, $J_n''(0)$, $\cdots$, $J_n^{xiv}(0)$.

Other major tables include tables of $e^{\pi x}$, $e^{-\pi x}$, $\sinh \pi x$, $\cosh \pi x$ by steps of 0.01 from $x=0$ to $x=10$; ten-place tables of $Y_0(x)$, $Y_1(x)$ by steps of 0.01 from $x=16$ to $x=25.50$; the numerical values of $Y_0'(x)$, $Y_0''(x)$, $\cdots$, $Y_0^{xiv}(x)$ for $x=16$ to $x=25$; the values of x^4, x^5 by steps of 1 from $x=1$ to $x=1000$; the values of x^{11}, x^{12} from $x=1$ to $x=100$; and ten-place tables of E, E' by steps of 0.001 for k^2. A few other smaller tables are included in the book.

The book is beautifully printed on paper of excellent quality. It should be of considerable value to all who work with Bessel functions.

J. B. SCARBOROUGH

Topologie. I. By C. Kuratowski. Warszawa-Lwow, Monografje Matematyczne, 1933. ix+285 pages.

This is the first volume of what the reviewer judges will be a three-volume treatise covering the whole field of topology. Vast though this undertaking may be, its success is completely assured by the amazingly masterful way in which the foundations are laid in this first volume. The object clearly is the presentation of all the major conclusions of topology in their proper logical sequence and at the same time in their simplest form and greatest generality. The degree of success attained by the author in this direction is indeed the most remarkable feature of the book; and after reading this volume, little doubt can remain but that when the project is completed it will find a central place among those works of mathematical literature which are of constant and lasting value.

The treatment is based on the undefined notion of the closure of a set (that is, the operation $\overline{A}$ of taking the set A together with all its points of accumulation) and on a system of axioms stated in terms of this notion. The organization of the material is such that the space becomes progressively more and more specialized as new axioms are added to the system. The first chapter deals with fundamental notions in what might be called general topological space, the second with separable metric spaces, and the third with complete spaces. The same plan is to be continued in the succeeding volumes where compact spaces, connected spaces, etc., will be treated. The methods of proof used in this volume are purely set-theoretic as distinguished from the combinatorial methods, which are not employed. The notation and symbolism used is so efficient that all proofs may be given in small space and they take on a decided "analytical" appearance. The brevity of the proofs, however, is more properly attributed to the ingenuity of the author and his excellent logical organization of the material.

After a brief introduction dealing with the fundamental operations of logic and of the theory of sets and with the notions of the cartesian product of sets and of functions or transformations defined on sets, the formal treatment begins in Chapter 1 with three axioms as follows: (I) $\overline{X+Y}=\overline{X}+\overline{Y}$, (II) if X is vacuous or a single point, $\overline{X}=X$, (III) $\overline{\overline{X}}=\overline{X}$. Upon this basis the author treats closed and open sets, boundary points and interior points, and neighborhoods of a point and localization of properties, dense sets, boundary sets, non-dense sets, sets of the first category, the property of Baire, development of sets in alternating series of closed sets, continuity of functions, and homeomorphisms. The author points out incidentally that these three axioms are equivalent to the first three axioms of Hausdorff.

Chapter 2, on separable metric spaces, is divided into five parts. The first of these is devoted to the introduction of the metric into the space, and in the course of the development two new axioms are added, axiom IV being an axiom of separation (normality) and axiom V an axiom giving a base for the space (perfect separability). Part B deals with problems concerning the power of sets. In part C there is presented practically the whole of the theory of dimension; and the fact that all of the principal results on this subject are here developed and proved in their full generality in less than 20 pages attests both the remarkable refinements which have been made in this theory within

the past five years and also the perfect logical arrangement of the material by the author. The fourth part deals with cartesian products and sequences of sets and the final part considers in great detail the Borel sets and functions measurable B in their most general form.

The third chapter, which has to do with complete spaces, treats questions relative to sequences of sets, extensions of functions, projective sets, analytic sets, totally imperfect sets, etc.

The value of the present volume lies not only in its usefulness as an introduction to the fundamentals of topology but more notably in the completeness with which the topics treated have been handled. Even the latest results are to be found here in what appears to be exactly their proper setting relative to the rest of the subject, and full advantage has been taken of even the most recently developed methods in refining the treatment.

G. T. WHYBURN

Symbiose, Parasitisme et Évolution. By V. A. Kostitzin. (Actualités Scientifiques et Industrielles, 96.) Paris, Hermann, 1934. 40 pp.

It is only very recently that the mutual influences of various species of plants and animals have been studied mathematically. Volterra's *Théorie Mathématique de la Lutte pour la Vie* (1931) was, of course, the pioneer treatise on the subject; and, aside from sections of Lotka's *Elements of Physical Biology* (1925), further discussion has been almost completely confined to periodicals.

Kostitzin's stimulating booklet is valuable, less because of new contributions to the mathematical methods than because of his enlargement of the scope of such studies,—particularly by a new emphasis on symbiosis (including commensalism). Even here—so complicated does the mathematics become for problems closely approximating to nature—the most difficult case considered is that of two species, each divided into two age-groups, with rates of birth and death dependent both on age and on the extent of the mutual aid. Kostitzin inquiries into periodic and stationary solutions of the differential equations, and into the stability of the latter type.

Equal emphasis is given to parasitism (harmful symbiosis). One section tells of the aid which mathematical analysis gave in determining how chlorogaster, a parasite of pagurians, develops in the hosts.

The treatment, in a booklet of this size, must too often be merely suggestive. Thus, we should be grateful for a thorough discussion of the remark that approximate agreement of periods usually signifies instability of a mechanical system, stability of a biological one.

A recent note by the author in the Paris Comptes Rendus (vol. 195, p. 1219) gives a suggestion of the wide applicability of this type of biomathematics. Therein he makes conjectures on the symbiotic periodicity of plant, animal, and inorganic matter of the earth as a whole; and on that of sediment, animal life, and H_2S in the Black Sea.

As this book shows, the way is open for a very salutary symbiosis (at the Christmas season we may prefer the term "commensalism") of mathematics and biology.

E. S. ALLEN

NOTES

In addition to the members of the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: A. A. Albert, H. Bateman, E. T. Bell, A. A. Bennett, B. A. Bernstein, Henry Blumberg, W. H. Bussey, R. D. Carmichael, E. W. Chittenden, Alonzo Church, A. B. Coble, L. E. Dickson, E. L. Dodd, Jesse Douglas, Arnold Dresden, Arnold Emch, H. T. Engstrom, G. C. Evans, Raymond Garver, Lois W. Griffiths, Morris Halperin, Olive C. Hazlett, Einar Hille, T. R. Hollcroft, E. V. Huntington, Louis Ingold, Dunham Jackson, Edward Kasner, A. J. Kempner, E. P. Lane, C. G. Latimer, N. H. McCoy, C. C. MacDuffee, W. D. MacMillan, W. A. Manning, A. D. Michal, H. H. Mitchell, C. N. Moore, F. D. Murnaghan, A. Nadai, F. S. Nowlan, Oystein Ore, H. B. Phillips, Tibor Radó, G. Y. Rainich, J. F. Ritt, W. E. Roth, J. A. Shohat, Virgil Snyder, M. H. Stone, J. D. Tamarkin, J. M. Thomas, J. H. M. Wedderburn, Marie J. Weiss, H. S. White, G. T. Whyburn, W. M. Whyburn, Norbert Wiener, W. A. Wilson, R. M. Winger, B. C. Wong. The editors desire publicly to acknowledge this service.

The first volume of a new journal, Meddelanden från Lunds Universitets Matematiska Seminarium, has appeared. It contains articles by Marcel Riesz and others.

The University of Illinois plans to publish the complete works of Professor G. A. Miller, who retired from its mathematical faculty recently. It is expected that six volumes of 500 pages each will be needed, and that the first will appear in 1935.

The fourth International Congress for Applied Mechanics was held on July 3–9, 1934, at Cambridge, England, under the presidency of Professor C. E. Inglis. About 320 persons attended, including representatives from 23 countries. More than 140 papers were read, in four sections dealing respectively with general mechanics, hydrodynamics, strength and properties of materials, water waves. Besides these sectional papers, the following delivered general lectures: V. Bush, on *Analyzing machines;* A. Caquot, on *The elastic limit in isotropic materials;* J. P. den Hartog, on *Vibrations in engineering;* T. von Karman, on *Turbulence;* E. Schmidt, on *Heat transference;* G. I. Taylor, on *The strength of crystals;* H. Wagner, on *Bodies gliding on the surface of water*. The next Congress will be held at Cambridge, Massachusetts, in 1938.

The British Association for the Advancement of Science will hold its next meeting at Norwich, September 4–11, 1935, under the presidency of Dr. W. W. Watts.

The eighth Scandinavian Mathematical Congress was held at Stockholm, September 14–18, 1934.

The committee on astronomy, applied mathematics, and physics of the National Research Council of Italy announces the following subject for a prize memoir: Concrete application of mathematical methods to the study of physical phenomena or technical processes into which phenomena of heredity enter. Competition closes March 30, 1935, and is restricted to citizens of Italy.

Professor C. E. Weatherburn, of the University of Western Australia, has been awarded the Hector medal and prize by the Royal Society of New Zealand, for his contributions to differential geometry.

The University Award granted by Rutgers University for distinguished services, was presented on September 18, 1934 to Associate Professor A. E. Meder of the New Jersey College for Women.

Yeshiva College, New York City, has conferred an honorary doctorate of humane letters on Professor Albert Einstein, of the Institute for Advanced Study. Professor Einstein was presented for his degree by David Eugene Smith, professor emeritus of the history of mathematics at Columbia University, who is a trustee of Yeshiva College.

Lehigh University has conferred an honorary doctorate on Dr. Irving Langmuir, of the General Electric Company.

On the occasion of the annual meeting of the Physico-Mathematical Society of Japan at Tokyo, April 2, 1934, Professor G. D. Birkhoff, of Harvard University, gave an address on some mathematical aspects of quantum mechanics. He also delivered a series of lectures on quantum mechanics at the National Tsing Hua University, Peiping, and the National University of Peiping, April 20–May 2; these lectures will appear in book form under the auspices of the two universities. On June 18–19, he gave two lectures at the Higher Normal School of Pisa, on the restricted problem of three bodies, which will appear in the Annali of that institution.

Professor D. J. Struik, of the Massachusetts Institute of Technology, delivered a series of lectures on differential geometry and on probability at the National Academy of Sciences "Antonio Alzate," in Mexico City, in July, 1934, on the occasion of the celebration of the fiftieth anniversary of that Academy. Professor Struik was elected to honorary membership in the Academy.

The retirement is announced of Professor H. S. Carslaw, for thirty-two years professor of pure and applied mathematics at the University of Sydney.

Professor Helmut Hasse, of Marburg, and Dr. Erhard Tornier, of Kiel, have been made joint directors of the Mathematisches Institut at Göttingen.

Mr. K. Mitchell has been appointed assistant lecturer in applied mathematics at the University of Leeds.

At the University of Nebraska, Professor W. C. Brenke has been made chairman of the department of mathematics. The retiring chairman, Professor A. L. Candy, will continue his teaching in the department.

Dr. J. M. Clarkson, of Cornell University, has been appointed to an assistant professorship at the North Carolina State College, Raleigh.

Dr. A. T. Craig has been promoted to an assistant professorship at the University of Iowa.

Professor George Gamow, head of the department of physics and mathematics at the Polytechnic Institute of Leningrad, has been appointed visiting professor at George Washington University for the present academic year.

Associate Professor A. H. S. Gillson has been promoted to a professorship at McGill University.

Dr. G. D. Gore has been promoted to a professorship at the Central Y.M.C.A. College, Chicago.

Dr. Paul Hemke has been appointed professor and head of the department of aeronautical engineering at the Rensselaer Polytechnic Institute.

Dr. R. C. Hildner has been appointed substitute professor of mathematics and astronomy at Mt. Union College, Alliance, Ohio.

Mr. L. L. Lowenstein has been appointed technical assistant to the Agricultural Adjustment Administration, in New York City.

Dr. W. I. Miller has been promoted to an assistant professorship at the University of Pittsburgh.

Dr. Gordon Pall has been promoted to an assistant professorship at McGill University.

Dr. H. H. Pixley has been appointed to an assistant professorship at Wayne University, Detroit.

Assistant Professor I. R. Pounder has been promoted to a professorship at the University of Toronto.

Dr. R. F. Rinehart, of Ohio State University, has been appointed professor of mathematics at Ashland College, Ashland, Ohio.

Dr. W. J. Robinson, of Ohio State University, has been appointed assistant professor of mathematics and chemistry at Washington College, Chestertown, Md.

Dr. C. F. Roos, chief of the research and planning division of the NRA, has been appointed professor of economics at Colorado College.

Professor G. Szegö, of Königsberg, has been appointed visiting professor of mathematics at Washington University, St. Louis.

Dr. H. L. Turrittin, of the University of Wisconsin, has been appointed adjunct professor at the College of Mines and Metallurgy, El Paso, Texas.

Professor R. L. Wilder has been appointed associate professor of mathematics at Johns Hopkins University, beginning September, 1935.

Associate Professor W. L. G. Williams has been promoted to a professorship at McGill University.

Assistant Professor E. W. Woolard, of George Washington University, has been appointed senior meteorologist at the United States Weather Bureau.

The following appointments to instructorships are announced: Albany College (Portland, Oregon): Mr. C. A. Keeler; Brooklyn College: Dr. Samuel Borofsky, Mr. Vincent Dillon (tutor); University of Buffalo: Dr. E. R. Ott; Case School of Applied Science: Dr. C. C. Torrance; College of the City of New York: Dr. Selby Robinson; Columbia University: Mr. George Comenetz; Duke University: Dr. F. G. Dressel; Georgia School of Technology: Dr. D. H. Ballou; Gooding College (Wesleyan, Idaho): Dr. Mary E. Haller; Harvard University: Dr. L. B. Andrews (in astronomy); Johns Hopkins University: Dr. E. K. Haviland; Lehigh University: Dr. D. H. Lehmer; University of Michigan: Dr. H. S. Kaltenborn; Mills College: Dr. Marjorie Leffler; New Jersey State Teachers College (Montclair): Dr. E. H. C. Hildebrandt; Rensselaer Polytechnic Institute: Dr. D. B. Ames; Rice Institute: Dr. E. F. Beckenbach; Rockhurst College: Mr. B. R. Wicker; College of St. Francis, Joliet, Illinois: Dr. F. C. Smith; Wayne University: Dr. C. H. Fischer; Westmoreland College (San Antonio, Texas): Dr. P. K. Rees.

Mr. Hermann Glauert, head of the Aerodynamics Department of the Royal Aircraft Establishment, died August 4, 1934, at the age of forty-one.

Professor Hans Hahn, of the University of Vienna, died July 25, 1934, at the age of fifty-four.

We are informed that the announcement in the November, 1933, number of this Bulletin that Professor Lichtenstein's death was by suicide was an error; we learn that he died of heart failure. We deeply regret our error, and beg to extend our apologies to the family of this distinguished mathematician.

The death is announced of Professor W. Franz Meyer, of the University of Königsberg, at the age of seventy-seven.

Professor W. H. Echols, of the University of Virginia, died September 25, 1934, at the age of seventy-four. Professor Echols had been a member of this Society since 1891.

The death is announced of Professor F. W. Hanawalt, of the College of Puget Sound. Professor Hanawalt had been a member of this Society since 1901.

Dr. William Benjamin Smith, professor emeritus of philosophy at Tulane University, died August 6, 1934, at the age of eighty-three. Professor Smith had been a member of this Society since 1891.

Associate Professor V. B. Teach of the Armour Institute of Technology died September 8, 1934.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

310. Miss M. G. Humphreys: *The representation of integers as sums of values of quartic polynomials.* Preliminary report.

The most general polynomial of degree r in x which represents integers for $x \geqq 0$ is of the form $P_r(x) = a_r {}_xC_r + a_{r-1} {}_xC_{r-1} + \cdots + a_1 {}_xC_1 + a_0$. When $r=4$ and $(a_4, \cdots, a_1)=1$, $a_0=0$, $a_4>0$, any sufficiently large integer is the sum of twenty-one values of $P_4(x)$ in all cases except when the coefficients satisfy one of three sets of conditions, of which the following is an example: $3/(a_4, a_3)$, $9+a_4, a_2 \equiv 6a_1 \pmod 9$, $a_4 \equiv 6a_2 \pmod 9$. The method used is similar to that of E. Landau in *Zum Waringschen Problem, Dritte Abhandlung* (Mathematische Zeitschrift, vol. 32 (1930), pp. 699–702). (Received August 7, 1934.)

311. Mr. J. F. Randolph: *Carathéodory measure and a generalization of the Gauss-Green lemma.*

The Gauss-Green lemma for the plane connects the double integral of a partial derivative of a function over a region R with the line integral of the function around the curve C bounding R. In the past many investigations have been concerned with the kind of regions and boundaries for which the lemma is valid. With the exception of a paper by Schauder, the boundary has been assumed to be a curve with a tangent almost everywhere. The present paper contains what seems to be extreme simplification of the conditions on the boundary. The Gauss-Green lemma is shown to hold for any simply connected region whose boundary has Carathéodory linear measure finite. Then by methods which have the effect of the usual cross cut scheme, applicable regions are extended to a wide class not simply connected. In the new auxiliary material is included the fundamental theorem that the inner Carathéodory linear measure of a set is the upper limit of the Carathéodory linear measure of closed components of the set. (Received September 26, 1934.)

312. Mr. Fritz Herzog: *Systems of algebraic mixed difference equations.*

The decomposition theory for systems of algebraic differential equations, developed by J. F. Ritt in his Colloquium Publication, and of algebraic difference equations, developed by Ritt and Doob, suggested a similar investigation

in the case of algebraic mixed difference equations. The author proves the result, analogous to that obtained in the two cases mentioned above, that a system of algebraic mixed difference equations can be decomposed in essentially one way into a finite set of irreducible systems. (Received September 27, 1934.)

313. Dr. A. B. Brown: *Functional dependence.*

Consider m functions of n variables and p parameters, with matrix of first partial derivatives of rank $<m$. Nowhere do we demand that a largest minor $\neq 0$ is not zero at a particular point. We prove, in the real case, that there is a functional relation in the large if the functions are sufficiently differentiable, generalizing a result of K. Knopp and R. Schmidt (1926), whose proof for their case $n \leqq m$ admits no obvious extension to our case. If $m=2$ and the functions are analytic (in complex space) we obtain an analytic relation in the small, generalizing a result of Bliss (1913), who treated the case $m=n=2$. Our proof is different from his. For general m, n, p and analytic functions of complex variables, Osgood showed (1916) that no analytic relation need exist. We prove that a property holds which would be valid if there were an analytic relation, namely, that the values of the given functions form a set which is nowhere dense in their space and cannot disconnect any region. (Received September 24, 1934.)

314. Dr. S. E. Warschawski: *On the higher derivatives at the boundary in conformal mapping.*

Let $w=f(z)$ map the circle $|z|<1$ conformally on the interior of a closed Jordan curve C. Conditions for the existence of $d^n f(z)/dz^n$ at the boundary, similar to those obtained in a previous paper for the case $n=1$, are obtained for every $n>1$. Let $\Theta(s)$ be an inclination angle of the tangent line (s=arc length). Set $\kappa^{(n)}(s)=d^n\Theta(s)/ds^n$. If C has "L-curvature" at $w_1=f(s_1)$, $s=s_1$, of order $n-1$, i.e., if $\lim_{s,s'\to s_1}[\kappa^{(n-2)}(s)-\kappa^{(n-2)}(s')]/(s-s')$ exists, and if (*) $\int_0^a |\kappa^{(n-2)}(s+t)+\kappa^{(n-2)}(s-t)-2\kappa^{(n-2)}(s)|\,dt/t^2$ converges, then $f^{(n-1)}(z)$ assumes continuous boundary values on $|z|=1$ near $z=z_1$ and is differentiable at $z=z_1$. If $\kappa^{(n-1)}(s)$ is continuous on C and (*) approaches zero uniformly on C with a, then $f^{(n)}(z)$ is continuous in $|z|\leqq 1$. It is shown how "the modulus of continuity" of $f^{(n)}(z)$ on $|z|=1$ depends on $\kappa^{(n-1)}(s)$. For every $n\geqq 1$, conditions are obtained under which $f^{(n)}(z)$ varies continuously in $|z|\leqq 1$ under a suitable deformation of C. (Received September 29, 1934.)

315. Dr. I. J. Schoenberg: *Extensions of theorems of Descartes and Laguerre to the complex domain.*

Let $f(x)=a_0+a_1x+\cdots+a_nx^n=0$ be an algebraic equation with real or complex coefficients. Mark the points a_ν in the complex a-plane. Draw in this plane through its origin any two lines Δ, Δ' defining two opposite angles A and B of common aperture ψ ($0<\psi\leqq\pi$) such that none of the points a_ν shall lie in the interior of either A or B. Call C and D the remaining opposite angles (of aperture $\pi-\psi$) defined by Δ and Δ', which contain all the points a_ν in their interior or their boundary. Call $v_a(A, B)$ the number of "variations" from the

domain C to the domain D or vice-versa in the sequence of coefficients a_0, $a_1, \cdots, a_n$, where as usual vanishing coefficients are to be disregarded. Then $v_n(A, B)$ is an upper bound for the number of zeros of $f(x)$ within the angular domain: $|\arg x| < \psi/n$. For real equations we can take $\psi = \pi$ and our result reduces to the extension of Descartes' rule given by N. Obreschkoff (Comptes Rendus, Paris, vol. 177 (1923), pp. 102–104). Rotating the x-plane by writing $x = e^{i\theta}z$, the above result gives information about the distribution of the arguments of all the zeros of $f(x)$ from the arguments of the coefficients of $f(x)$ only. Extensions to a class of exponential sums are given. Furthermore it is shown that a theorem of Laguerre (Oeuvres, vol. 1, p. 41) actually applies to the roots in certain half-planes and circles of the complex plane. Cauchy's index theorem supplies the proofs. (Received September 29, 1934.)

316. Dr. R. H. Cameron (National Research Fellow): *Linear differential equations with almost periodic coefficients.*

A vector function of the real variable t is said to be of the almost periodic form if it can be expressed as a finite sum of terms each of which is the product of an exponential function, a non-negative integer power of t, and an almost periodic vector function. This paper gives certain sets of necessary and sufficient conditions that all of the solutions of a linear system (which may or may not be homogeneous) of ordinary differential equations with almost periodic coefficients should be of the almost periodic form. The paper also gives certain sufficient conditions that a particular solution be of the almost periodic form. (Received September 27, 1934.)

317. Mr. Walter Prenowitz: *The characterization of plane collineations.*

Let R be a region of the euclidean plane. A set of line intervals contained in R is called a family of lines in R, if the ends of each interval of the set are not in R, and each point of R is on exactly one interval of the set. If four families of lines are contained in R and no two have a line in common they are said to form a 4-web in R. The principal result of this paper is as follows: Any topological transformation of region R, which carries a 4-web of lines in R into a 4-web of lines, is projective. This was proved by E. Kasner (this Bulletin, vol. 9, pp. 545–546) on the assumption that the transformation is doubly differentiable. For collineations on the projective plane, the following theorem is proved: A one-to-one point transformation on the projective plane is a collineation, if it carries three independent pencils and a line not of these pencils into three pencils and a line, respectively. This is an improvement on the result stated by the author in a preliminary report (this Bulletin, vol. 38, p. 345). The author has also derived several theorems related to this, including an analogue for a region of the euclidean plane. (Received October 1, 1934.)

318. Dr. E. J. Finan: *On the number theory of certain non-maximal domains of integrity of the total matric algebra of order* 4.

The following four matrices $\|a_{ij}\|$, $(i, j = 1, 2)$, form a basis for a non-maximal domain of integrity of the total matric algebra of order 4: In the first,

$a_{11}=1$, $a_{12}=a_{21}=a_{22}=0$; in the second, $a_{12}=1$, $a_{11}=a_{21}=a_{22}=0$; in the third, $a_{22}=1$, $a_{11}=a_{12}=a_{21}=0$; in the fourth, $a_{21}=k$, $a_{11}=a_{12}=a_{22}=0$, where k is a rational prime. It is shown that if n is any number of the domain a necessary and sufficient condition that n be a prime is that the determinant of n be a rational prime. Ideals of the domain are represented by integral matrices of order 4. This approach is due to C. C. MacDuffee. The class number of the domain is proved to be 3. (Received September 24, 1934.)

319. Dr. James Singer: *A theorem in finite projective geometry and a number theory application.*

It is well known that a finite projective plane, $PG(2, p^n)$, exists for every prime p and positive integer n. There are p^n+1 points on a line and $p^{2n}+p^n+1$ points and lines in the plane. Furthermore, if there are $m+1$ points on a line of a finite projective plane, m is a power of a prime. We prove first that in any finite projective plane there exists a collineation which carries a point P into a point P', P' into P'', $\cdots$, $P^{(q)}$ into P, where $q=p^{2n}+p^n$ and the P's are all distinct. We then show that the points and lines of the plane can be exhibited in a regular array, that is, a matric array of p^n+1 rows and $p^{2n}+p^n+1$ columns whose elements are symbols for points, whose columns represent the points on a line, and each of whose rows is a cyclic permutation of the first row. The existence of the regular array enables us to prove this interesting theorem: A necessary and sufficient condition for the existence of $m+1$ integers d_0, d_1, $\cdots$, d_m such that their m^2+m differences $d_i-d_j (i\neq j)$ be congruent modulo m^2+m+1 to the integers $1, 2, \cdots, m^2+m$ in some order is that m be a power of a prime. (Received September 24, 1934.)

320. Dr. J. S. Frame: *On the numerical determination of tables of characteristics of finite groups.*

Between the elements of the square array, giving by rows and columns the characteristics of matrices, one from each conjugate set in each irreducible linear representation of a finite group, there exist many numerical relations valid for all finite groups. Further relations or restrictions on this table obtain for simple groups. In this paper six of the former relations and five of the latter restrictions are shown to determine uniquely a single table corresponding to each of the group orders 60 and 168. No inner structural properties of the particular group are used, except that the group is simple and has the given order. The tables so obtained are those of known simple groups. (Received October 1, 1934.)

321. Dr. Leo Zippin: *Compact zero-dimensional abelian groups.*

A complete invariant system $I(X)$ of a compact zero-dimensional group X is given, such that X and Y are isomorphic if and only if $I(X)=I(Y)$. This equivalence-problem was solved indirectly by J. W. Alexander (*On the homology groups of abstract spaces*, Annals of Mathematics, vol. 35 (1934)) by reference to the (discrete) character groups associated with them. This proof is independent of such considerations. It is shown, further, that if $I(X)=I(Y)$

then certain subgroups of a very general nature X' and Y' (of X and Y respectively) are isomorphic, and any isomorphism between them can be extended to the original groups. (Received September 29, 1934.)

322. Professor C. N. Moore: *On criteria for Fourier constants of L-integrable functions of several variables.*

In a paper presented at the last annual meeting of the Society (see this Bulletin, abstract 40-1-41) the author has given a criterion for the Fourier cos-cos coefficients of an L-integrable function of two variables. In the present paper a more general criterion, which includes the previous one and other related criteria, is given. Furthermore the theorem is extended to functions of more than two variables and the related multiple Fourier series. The results obtained constitute a generalization to multiple series of a criterion for simple series recently given by Cesari (see Bollettino della Unione Matematica Italiana, vol. 13 (1934), p. 100). (Received September 28, 1934.)

323. Mr. Garrett Birkhoff: *Lattices of equivalence relations.*

It is shown that the possible definitions of equivalence within a class of n objects constitute a lattice E_n, dually isomorphic with the lattice of the subalgebras of a Boolean algebra of order 2^n. E_n occupies in lattice theory a position like that of the symmetric group on n letters in group theory; the parallel extends to the group of automorphisms of E_n, the classification of similar elements, transitivity, order, etc. Moreover E_n is "simple" in the sense of having no proper homeomorphic image. Various abstract methods for combining lattices are discussed, and the sublattices of E_4 are enumerated. Finally, it is shown that every lattice of subgroups is isomorphic with a lattice of equivalence relations, and conversely. (Received October 1, 1934.)

324. Professor Oystein Ore: *On the foundation of abstract algebra.*

This paper contains an attempt to unify the various branches of abstract algebra. By means of the theory of structures one can obtain a basis for such a general theory including group theory, ideal theory, moduli, etc., and the various decomposition theorems of these theories may be derived from a common source. (Received September 26, 1934.)

325. Professor Harold Hotelling: *On correlation between vectors, and a function associated with the tetrad difference.*

A generalization of the correlation coefficient is introduced which is appropriate for measuring the relationship between a set of s and a set of t variates, or between a random vector in s and one in t dimensions. This vector correlation coefficient q is invariant under internal linear transformations of the variates of either set, lies between -1 and $+1$, and for $s=t$ is actually the correlation between the corresponding s-rowed determinants of the two matrices of observations. Its distribution in samples from normal populations involves s parameters, if $s \leqq t$. The most important case is that in

which $s-1$ of the parameters vanish; for this case the sampling distribution of q is determined exactly, and expressions for the moments are obtained. For $s=t=2$, the vector correlation takes the form $q=(r_{13}r_{24}-r_{14}r_{23})(1-r_{12}^2)^{-1/2}\cdot(1-r_{34}^2)^{-1/2}$. The distribution provides exact probabilities, valid even for small samples, for the tetrad difference in the numerator. Such a test of significance has previously been the object of extensive investigation by means of rough asymptotic approximations to the standard error, and is needed in mathematical psychology. (Received September 11, 1934.)

326. Dr. J. L. Doob: *Note on probability.*

The von Mises theory has as its fundamental element the idea of a collective: a sequence of observations on which two conditions are imposed. The second of these conditions states the impossibility of a system in a game of chance. Using this condition von Mises has shown that a collective, considered as a sequence of mathematical elements, cannot be obtained by analytic processes, and cannot be fully specified. Such a definition has been criticized by Kamke in a recent paper. To avoid this difficulty a modification is made in the two conditions of von Mises to make them not conditions on a fixed sequence of observations but conditions on the mechanism of the experiment. The two conditions of this modified theory correspond to theorems in the classical theory. The first condition corresponds to the strong law of large numbers. The second condition corresponds to the theorem that a system is impossible in a game of chance. A proof is given of this fact, in terms of the classical theory. By means of these considerations, an analysis is made of the relations between the von Mises theory, the modified theory, and the classical theory. (Received September 26, 1934.)

327. Mr. Garrett Birkhoff: *Linear extension of complex-valued linear functions in Banach space.*

It is shown that it is possible to define a linear complex-valued function over a plane in a three-dimensional normed vector-space, which cannot be extended without increasing the modulus. (Received September 14, 1934.)

328. Mr. Garrett Birkhoff: *On monogenic extension of linear functionals.*

It is shown that any monogenic linear functional of modulus M defined over a linear manifold of a Banach space B in which the scalars are complex numbers can be extended over B to a monogenic linear functional of modulus $\leqq 2^{1/2}M$. (Received September 24, 1934.)

329. Mr. Garrett Birkhoff: *Orthogonality in normed vector spaces.*

It is proved that if in a normed vector space of three or more dimensions, the norm is differentiable, and the relation of perpendicularity is reciprocal, then the norm is Pythagorean. Nevertheless there exist infinitely many non-

isometric normed vector spaces of two dimensions which are "orthogonal." (Received September 14, 1934.)

330. Professor H. V. Craig: *On a generalized tangent vector.* Paper II.

By means of the vectors introduced in Paper I a metric tensor and a connection are constructed, such that the intrinsic derivative of the metric tensor is zero while the autoparallel curves are the extremals associated with F. The arguments of F are the coordinate variables and their derivatives up to order m. (Received September 28, 1934.)

331. Mr. R. M. Robinson: *The Bloch constant $\mathfrak{A}$ for a schlicht function.*

Consider the functions $f(x)$ which are regular within the unit circle, for which $f'(0)=1$, and which map the unit circle on a plane region. Then it is known that a circle of radius 9/16 can be found within the map. This is one of a number of results in a paper by Landau (Mathematische Zeitschrift, vol. 30 (1929), pp. 608–634). The upper bound of the constants which can replace 9/16 in this statement defines the constant $\mathfrak{A}$, so that $\mathfrak{A} \geq 9/16$. From above the best inequality known was $\mathfrak{A} \leq \pi/4$, which was given by Szegö. By mapping the unit circle on a circle with radial slits, the author obtains the following results: using a single slit extending to the origin, $\mathfrak{A} < 3/4 < \pi/4$; using three slits extending half way to the origin, $\mathfrak{A} < 11/16$; using four or six slits suitably chosen, $\mathfrak{A} < 2/3$. (Received September 24, 1934.)

332. Professor G. T. Whyburn: *Concerning curves of finite degree and local separating points.*

In this paper a study is made of curves of finite degree in the sense of H. Kamiya (see Tôhoku Mathematical Journal, vol. 34 (1931), pp. 58–72) in connection with their local separating points. It is shown that a compact continuum K will be of finite degree if and only if every subcontinuum of K contains uncountably many local separating points of K. The property of being of finite degree is cyclicly extensible and reducible, and any continuum admits a *finest* upper semi-continuous decomposition so that the hyperspace is a curve of finite degree. Also it is shown that, whereas in any curve of Menger-Urysohn order ≤ 3, *degree* and *order* are identical for all points, there exists a curve of order 4 which is of infinite degree at every point. (Received September 26, 1934.)

333. Professor G. T. Whyburn: *A decomposition theorem for continua.*

Let P be any local property of a continuum such that the set of all non-P points of any compact continuum is either vacuous or of dimension >0. The theorem is proved that if K denotes the set of all non-P points of a compact continuum M and M is decomposed upper semi-continuously into the components of $\overline{K}$ and the points of $M-\overline{K}$, then every point of the hyperspace H

is a P-point of H. Taking the properties of being (i) locally connected, (ii) regular, (iii) rational, (iv) of dimension $<n$, or (v) of belonging to no continuum of convergence, or (vi) of belonging to no continuum of condensation for P, we obtain that the hyperspace is (i) locally connected, (ii) a regular curve, (iii) a rational curve, (iv) of dimension $<n$, (v) hereditarily locally connected or (vi) without continua of condensation, respectively. Number (i) is the well known result of R. L. Moore concerning the decomposition of a continuum into its prime parts. (Received September 26, 1934.)

334. Professor H. S. White: *The ten common secants of two twisted cubic curves.*

These have been enumerated by Cremona on degenerate curves, by Halphen with the use of metric auxiliaries, and by Segre employing a quadric locus in five-space. The author finds a simple way to reach the enumeration, with two others of some interest, by binary correspondences in three-space and generic curves in random position, the whole process being projective. (Received October 9, 1934.)

335. Mr. Max Zorn: *On infinite algebra.*

Instead of the well-ordering principle a certain inductive principle equivalent to it is introduced and one obtains a considerable simplification in the proofs of Steinitz' theorems on transfinite algebra. (Received October 3, 1934.)

336. Mr. W. E. Sewell: *Generalized derivatives and approximation by polynomials.*

Let $f(z)$ be analytic in the simply connected region R bounded by the analytic Jordan curve C and continuous in the corresponding closed region; then Riemann's definition of a generalized derivative on the axis of reals is extended to $f(z)$ on C. The existence of a generalized derivative under conformal transformation is preserved. Bernstein's theorem on the modulus of the derivative of a polynomial is generalized. Sufficient conditions that $f(z)$ have a generalized derivative are established in terms of the degree of approximation of polynomials and also in this connection more general point sets are considered. The relation between Hölder conditions and generalized derivatives is considered in connection with approximation. An example of the results established is the following: Theorem. *If for every n there exists a polynomial $P_n(z)$ of degree n, such that $|f(z)-P_n(z)|<M/n^\alpha$, $\alpha>0$, on and within an analytic Jordan curve C, and $f(z)$ is analytic interior to C, then $f(z)$ has derivatives of all orders α' less than α on the curve C.* (Received September 11, 1934.)

337. Mr. Garrett Birkhoff: *Operational identities on point sets.*

There exist operational identities which perfectly characterize systems of point sets closed relative to (i) finite sums and products, (ii) complements and finite sums and products. This note shows that no such operational identities

can exist relative to (iii) enumerable sums and products, (iv) transfinite sums and products, (v) complements and enumerable sums and products. (Received October 3, 1934.)

338. Dr. Solomon Kullback: *On the Bernoulli distribution.*

By employing merely the law of combination of probabilities we are enabled to derive in an elementary manner, without the use of characteristic functions or the explicit form of the distribution itself, the expected values of the moments of the Bernoulli distribution. The discussion is readily extended to the Poisson exponential distribution and to distributions of the Lexis and Poisson type. (Received October 11, 1934.)

339. Mr. Garrett Birkhoff: *Ideals in algebraic rings.*

The main result of the paper is as follows: Let x be any algebraic integer, $f(x)=0$ the irreducible equation for x, and $R(x)$ the ring generated by x and the rational integers. Then the conditions that the ideals of $R(x)$—except zero—should (1) satisfy the distributive law relative to g.c.f. and l.c.m., (2) be "regular" in the sense of Grobner, and (3) be of the canonical type described by E. Noether, are the same, and are (4) that if $g(x)$ is any irreducible factor of $f(x)$ modulo any prime p, then either $g^2(x)$ is not a divisor of $f(x)$ modulo p, or else the g.c.f. of the ideals $g(x)R$ and p^2R divides pR. (Received October 17, 1934.)

340. Professor R. S. Burington: *On the equivalence of quadrics in m-affine n-space and its relation to the equivalence of 2m-pole networks.*

The work of Cauer (Göttinger Nachrichten, 1934) and others on the equivalence of $2m$-pole electrical networks indicates the importance of the matric study of quadratic forms F under m-affine non-singular transformations, $x_i=x_i'$, $i=1, \cdots, m$, $x_j=\sum_{k=1}^{n} b_{jk} \cdot x_k'$, $j=m+1, \cdots, n$, of matrix T. Two symmetric matrices A and B, of forms F and G, are m-affine congruent if and only if there exists a T such that $A=T' \cdot B \cdot T$. It is assumed that the elements of the above matrices belong to a field. A set of integer, absolute, and matric invariants is exhibited in terms of which necessary and sufficient conditions for the m-affine congruence of A and B are given. A detailed reduction for the 4-pole case is given; $m=0$ yields the projective theory, $m=1$, the usual affine case. (See R. S. Burington, American Mathematical Monthly, vol. 39 (1932), pp. 522–32; Physical Review, vol. 45, p. 429.) The relation of this work with that of Cauer is discussed. (Received October 31, 1934.)

341. Professor S. S. Cairns: *Triangulation of the manifold of class* 1.

A construction is given for triangulating the manifold of class 1 as defined by Veblen and Whitehead in *The Foundations of Differential Geometry* (Cambridge Tract No. 29, 1932, Chapter VI). The method employed is a refinement of that developed in the writer's earlier paper on the triangulation of regular

loci (Annals of Mathematics, vol. 35 (1934), pp. 579–587). (Received November 7, 1934.)

342. Dr. L. A. Dye: *Space involutorial transformations of the Geiser and Bertini types.*

One form of generalization of a plane involution is a space involutorial transformation in which each plane of a pencil is invariant and in each such plane there is a plane involution of the same type. Transformations of the Geiser and Bertini types are discussed in this paper by means of a mapping on a cubic surface. The signature of the Bertini transformation is $I_{120n+61}$: $l^{120n+24+6t}+(O,\ \overline{O})^{120n+40}+C^{6}_{12n+6}$; the Geiser transformation has the signature I_{24n+19}: $l^{24n+11+3t}+O^{24n+14}+C^{3}_{12n+6}$. The fundamental points O, $\overline{O}$ lie on the fundamental line l which meets the curve C_{12n+6} $12n$ times. (Received November 7, 1934.)

343. Professor Orrin Frink: *Differentiation of sequences.*

Conditions under which it is permissible to differentiate term by term a convergent sequence of functions can be found in the literature, but they usually include the hypothesis that the derived sequence converges. Without this hypothesis, a number of conditions, one of them being the equicontinuity of the derived sequence, are here shown to be sufficient for termwise differentiability. Some of the conditions found are, like that of equicontinuity, connected with the question of the compactness of sets of functions for different types of convergence. (Received November 3, 1934.)

344. Dr. M. C. Hartley: *On the mapping of the* 24*-tuples of the involutorial* G_{24} *in a plane upon a quadric.*

The permutations of the quantities $\pm x_1$, $\pm x_2$, $\pm x_3$ considered as projective coordinates in a plane determine 24 points in the plane and hence an involution of 24-tuples which may be mapped on a rational surface. If ϕ_1, ϕ_2, ϕ_3 are the elementary symmetric functions of x_i^2, $y_i=a_i\phi_1^4+b_i\phi_1^2\phi_2+c_i\phi_1\phi_3+d_i\phi_2^2$ is a symmetric function of degree 8. There are then 4 linearly independent functions y_i $(i=1, 2, 3, 4)$ which may be set proportional to the 4 projective coordinates of a point in space. To every 24-tuple corresponds, in a (1, 1) manner, a point (y). The locus of these points (y) is proved to be a quadric cone, whose equation is found and discussed. The mapping of this cone upon the plane leads to properties of plane curves of order $8n$, and to theorems on octics. (Received November 7, 1934.)

345. Dr. M. R. Hestenes (National Research Fellow): *A basic theorem in the problems of Lagrange and Bolza.*

In the present paper we prove the following theorem: Let E_{12} be an extremal arc in the problem of Lagrange or Bolza in the calculus of variations. If E_{12} satisfies the strengthened Clebsch condition and has on it no point conjugate to its initial point 1, then E_{12} is an extremal of a Mayer field. With the help of this result one can obtain the classical sufficiency theorems for these problems

without assumptions regarding normality. In this sense the present paper completes the results recently obtained by the author for the general problem of Bolza (Transactions of this Society, October, 1934). The theorem here given has been established recently by Morse (Abstract 40-9-303) and by Reid (Abstract 40-9-306). The method here used is different from those of Morse and Reid and is of interest since we show that the theorem here given is an easy consequence of certain results established by the author for the general problem of Bolza. (Received October 24, 1934.)

346. Dr. Rosella Kanarik: *Fundamental regions in S_4 for the Hessian group.*

In this paper the fundamental regions for the group of 216 in three variables are determined. Nine Hermitian forms which form a single set of conjugates under the group are chosen. Points which do not lie on the 36 hypersurfaces obtained from these forms are considered. Instead of the expected 9! arrangements of values according to order of magnitude of the nine definite forms, only 19,008 possible arrangements can be found. The substitutions $x_2/x_1=re^{i\theta}$ and $x_3/x_1=se^{i\phi}$ are made in the forms. That the first form may be selected in any one of nine ways and the second in any one of eight ways is shown by the substitution of coordinates of points in the forms. The third form, however, can be selected in only six ways. The remaining forms give rise to only 44 arrangements. Thus, since every arrangement has 216 conjugates under the group, there are 216 fundamental regions composed of 88 subregions. (Received November 5, 1934.)

347. Professor N. H. McCoy: *On the rational canonical form of a function of a matrix.*

Let A be a given matrix of order n with elements in the field F of complex numbers, and $\phi(\lambda)$ a polynomial with coefficients in F. The problem of determining the elementary divisors of $\phi(A)$ from those of A has been discussed by Kreis, Krishnamurthy, Turnbull and Aitken, Rutherford and Amante. The purpose of the present paper is to solve the corresponding problem for the case in which the field F is replaced by an arbitrary field K. A brief application is given to the rational solution of scalar matric equations. (Received October 19, 1934.)

348. Professor E. G. Olds: *Distributions of greatest variates, least variates, and intervals of variation in samples from a rectangular universe.*

Neyman and Pearson have given the distribution of intervals of variation for samples from a continuous, rectangular universe. Rider has considered an infinite, rectangular population of discrete variates, and obtained the distribution of intervals of variation for samples of four from a ten-class universe. The present paper gives the distribution of intervals of variation for samples drawn, without replacement, from the population characterized by the frequency distribution $f(x)=1$ for $x=0, 1, 2, \cdots, b$, (and zero elsewhere), and

compares the distribution with those mentioned above. It also derives the distributions of greatest and of least variates, for samples from this same universe, and compares them with those obtained for a continuous universe. (Received November 1, 1934.)

349. Professor W. E. Roth: *On k-commutative matrices.*

(a) The matrix B_k, defined by the recurrence formula $B_{i+1}=AB_i-B_iA$, $(i=0, 1, \cdots, k-1)$, where A and $B=B_0$ are given $n\times n$ matrices, is the k-commute of A with respect to B. (b) The matrix, A, is k-commutative with respect to B if $B_k=0$ and $B_{k-1}\neq 0$. According to these definitions A and B are commutative in the usual sense if $B_1=0$; and quasi-commutative in the McCoy sense (*On quasi-commutative matrices*, Transactions of this Society, vol. 36 (1934), pp. 327–340) if A and B are mutually 2-commutative. Mutually k-commutative matrices, $k>2$, likewise exist. The present paper exposes certain properties of k-commutative matrices and of mutually k-commutative matrices and develops the algebra of such matrices. (Received November 1, 1934.)

350. Professor J. L. Synge: *Principal null-directions in space-time defined by an electromagnetic field.*

In a flat space with fundamental form $dx_r dx_r$ any symmetric tensor T_{rs} defines principal directions ξ_r and invariants X by means of the equations $T_{rs}\xi_r+X\xi_r=0$. The paper investigates these directions when T_{rs} is the energy tensor of an electromagnetic field. It is found that the four invariants are of the form $-k$, $+k$, $-k$, $+k$, where k differs from zero if and only if the field is not null, a null-field being one in which the electric and magnetic vectors are equal and perpendicular. The principal directions are partially indeterminate, but among them there exist two principal null-directions in general, degenerating to one when the field is null, and in that case represented to any observer by a point travelling along the pointing vector with the velocity of light. In the general case the two principal null-directions define a 2-flat, and if the time-axis is taken in it, the electric and magnetic vectors have a common line of action. (Received November 7, 1934.)

351. Professor J. L. Synge: *Some intrinsic and derived vectors in a Kawaguchi space.*

In a Kawaguchi space of order m the length of a curve $x^i=x^i(t)$ is $\int_{t_1}^{t_2} F dt$, where F is a function of the parameter t and of the coordinates and their derivatives with respect to t up to the mth order. It is shown that any curve defines a set of $m+1$ vectors $E_i^{\,p}=\sum_{q=p}^{m}(-1)^q{}_qC_p(d^{q-p}/dt^{q-p})(\partial F/\partial x^{(q)i})$, $(p=0, 1, \cdots, m)$, where $x^{(q)i}=d^qx^i/dt^q$: some of these have already been found by H. V. Craig. If $m=1$ (Finsler space), the two vectors of this type are well known: $E_i^{\,0}=F_{(0)i}-dF_{(1)i}/dt$, $E_i^{\,1}=-F_{(1)i}$. Other intrinsic vectors are also obtained. For an assigned vector field X^i, the invariant $X^iE_i^{\,p}$ is used as a "generating function" to obtain a set of covariant vectors involving X^i and its derivatives, $D_{ij}^{p,r}X^j=\sum_{p=r}^{2m-p}(-1)^q{}_qC_r(d^{q-r}/dt^{q-r})(X^jE_j^{\,p})_{(q)i}$, $(p=0, 1, \cdots, m;\ r=0, 1, \cdots, 2m-p)$. One of these formulas (slightly modified)

enables us to define parallel propagation in a Kawaguchi space. (Received November 7, 1934.)

352. Professor W. J. Trjitzinsky: *Laplace integrals and factorial series in the theory of linear differential and linear difference equations.*

In the fields of ordinary linear differential and linear difference equations, whose coefficients are convergent series in negative powers of $x^{1/p}$ (integer $p \geqq 1$) with, possibly, a few positive powers present, a fundamental problem is that concerning the nature of the solutions in the vicinity of the singular point ($x=\infty$). The method of Laplace integrals, leading to convergent factorial series developments (summation of formal series solutions), is not applicable to the unrestricted problem, the latter being treated completely by asymptotic methods in recent papers by G. D. Birkhoff and by the author (Acta Mathematica). However, the method of Laplace integrals, whenever applicable, is more satisfactory than the asymptotic method. In the present work application of the stated method is carried considerably beyond the results of earlier writers (notably, Nörlund and Horn). "Gegenbeispiele" are given making it evident that, in so far as the essential features of the theory are concerned, the main theorems established are substantially of the greatest possible generality. This work will appear in the Transactions of this Society. (Received November 7, 1934.)

353. Professor G. T. Whyburn: *Concerning K-perfect sets.*

Let K be any class of closed sets. A set A will be said to be K-perfect provided A is closed and $K(A)=A$, where $K(A)$ denotes the K-derivative of A as defined in my paper in the American Journal of Mathematics, vol. 54 (1932), p. 170. In this paper it is shown that if C is any separable and metric space, then for any class K, C is the sum of a K-perfect set and a countable number of K-sets. Also relations are established between the classes of all K-perfect sets corresponding to various choices of the class K. (Received November 9, 1934).

354. Professor R. L. Wilder: *A characterization of manifold boundaries in E_n dependent only on lower dimensional connectivities of the complement.*

This paper is a sequel to our recent paper *Generalized closed manifolds in n-space* (Annals of Mathematics, October, 1934), to which we refer for definitions of terms used. In $E_n (n \geqq 3)$, let a compact point set M be a common boundary of (at least) two domains D_1 and D_2 such that $D_k (k=1, 2)$ is uniformly locally i-connected for $0 \leqq i \leqq n_k$, where $n_1+n_2=n-3$. Then, if one of the numbers $p^{n_k+1}(D_k)$ is finite, M is a generalized closed $(n-1)$-manifold. This contains Principal Theorem E of our former paper as a special case (as well as Theorem 20, Mathematische Annalen, vol. 109, p. 305), showing, incidentally, that the restriction on $p^{i+2}(D), \cdots, p^{n-2}(D)$ in that theorem is unnecessary. The condition that a $p^{n_k+1}(D_k)$ be finite may be replaced by the

condition that for one D_k there is an $\epsilon>0$ such that (n_k+1)-cycles of D_k of diameter $<\epsilon$ bound in D_k. We also obtain extensive generalizations of Theorems 11 and 15 of our Mathematische Annalen paper referred to above. (Received November 9, 1934.)

355. Professor H. J. Ettlinger: *Linear derivative inequalities.*

By the method of the integrating factor, the first order linear inequality, $v'+P(x)v\leqq Q(x)$, where $v(x)$ is an absolutely continuous function on (0, 1), and $P(x)$, $Q(x)$ are integrable (Lebesgue) on (0, 1), is solved. The result for $Q(x)\equiv 0$, $v(0)=0$ leads immediately to establishing the uniqueness of an absolutely continuous solution for the general first order system satisfying a Lipschitz condition. The result for the non-homogeneous relation establishes at once the continuity of the solution of the general first order system as a function of the coefficients of the system and the set of initial values. (Received October 23, 1934.)

356. Mr. F. S. Harper: *On a function to represent infantile mortality.*

Many efforts have been made to increase the range of ages for which Makeham's function will satisfactorily represent the $\log l_x$, where l_x represents the number living at age x, because of certain theoretical and practical advantages principally in the computation of joint life functions. Difficulties are met at both extremes of the mortality table; however, the infant ages present the greater problem due to the very rapid decrease in l_x during the first year of life. In the present paper a function is introduced which represents $\log l_x$ by months in the first year of life with sufficient accuracy to reproduce the recorded values of l_x for one of the U. S. Life Tables with an error of less than 1 in 100,000. This function is then used in conjunction with Makeham's to reproduce l_x from age 0 to 81 with a maximum error of two per cent. An expression valid for the whole range of life has been found by introducing an auxiliary three parameter function to represent the error from age 81 to the end of the table. Formulas are developed to be used in the computation of joint life functions involving two lives. (Received October 24, 1934.)

357. Dr. Solomon Kullback: *On the multinomial distribution.*

The notions introduced in a previous paper, *On the Bernoulli distribution*, are applied to derive the expected values of the moments and product moments of the multinomial distribution. The results are very simply formulated in the symbolism of the calculus of finite differences. (Received October 26, 1934.)

358. Mr. G. D. Nichols: *The arithmetized expansions for certain doubly periodic functions of the third kind.*

The explicit arithmetized Fourier series developments are obtained for $\theta_1'^2\theta_\alpha(x+y)/\theta\beta^2(x)\theta_\gamma(y)$ and $\theta_1'^2\theta_\alpha^2(x+y)/\theta\beta^2(x)\theta_\gamma^2(y)$, where the θ's are the Jacobi theta functions, x and y are independent complex variables, and (α, β, γ) are a certain sixteen triads out of a possible sixty-four which can be

selected from 0, 1, 2, 3. The arithmetical results obtainable from these expansions involve incomplete numerical functions in two variables. Some of these results are given. (Received October 29, 1934.)

359. Mr. K. S. Ghent: *A generalization of Waring's theorem.* Preliminary report.

R. E. Huston (in his doctor's thesis, University of Chicago) has shown that the s used in the first Hardy-Littlewood theory suffices for the more general function $f=a_1x_1^k+\cdots+a_sx_s^k$ if certain congruential conditions are satisfied by the coefficients $a_1, \cdots, a_s$. It can now be shown that the smaller s of the second Hardy-Littlewood theory will suffice for the same function under the same congruential conditions. (Received October 16, 1934.)

360. Dr. M. S. Robertson (National Research Fellow): *Stieltjes integral for functions convex in one direction.*

In this paper are considered the class $\mathfrak{F}$ of analytic functions $f(z)=z+\sum_2^\infty a_nz^n$, a_n real, univalent, and *convex in the direction of the imaginary axis* for $|z|<1$, i.e., every straight line parallel to the imaginary axis cuts the image of the circle $|z|=r$, for every r in the interval (0, 1), in not more than two points. The author shows that the necessary and sufficient condition that $f(z)$ belong to class $\mathfrak{F}$ is that it can be represented in the form $f(z)=(1/4\pi i)\int_{-\pi}^{\pi}[\log(1+ze^{i\theta}/1+ze^{-i\theta})]d\alpha(\theta)/\sin\theta$ where $\alpha(\theta)$ is an odd, non-decreasing function of θ in $(-\pi, \pi)$ given by the series $\alpha(\theta)=\theta+\sum_1^\infty[(k+1)a_{k+1}-(k-1)a_{k-1}/k]\sin k\theta$. Various properties of $f(z)$ are deduced from this representation. (Received October 26, 1934.)

361. Professor A. T. Craig: *On the distribution of a certain product moment.*

In the present paper, the distribution function of the product moment $u=\sum x_iy_i/N\sigma_x\sigma_y$ is found when x_i and $y_i (j=1, 2, \cdots, N)$ are $N=2k$ values of two normally correlated variables x and y. If the coefficient of correlation between x and y is zero, the distribution function of u is identical with that of the arithmetic mean of samples of k independent items drawn from a population characterized by Laplace's first law of probability. (Cf. Mayr, Monatshefte für Mathematik und Physik (1920), p. 25). (Received November 1, 1934.)

362. Mr. A. P. Cowgill: *On the summability of a class of series of Jacobi polynomials.*

This paper proves that the series $\sum_{n=1}^\infty n^i[(p+1)(p+3)\cdots(p+2n-1)/2^nn!]\cdot X_n(p, (p+1)/2, (1-x)/2)$, $-1<p\leqq 1$, where X_n is a Jacobi polynomial and i is a positive integer, is summable (C, k), $k>i-\frac{1}{2}$, $-1<x<1$. Legendre polynomials, $p=1$, had previously been proved summable (C, k), $k>i-\frac{1}{2}$. In the proof the sum of n terms of the given series is transformed by the recursion formula for Jacobi polynomials into a new sum of n terms, plus four additional terms. Convergence factors for summability $(C, i-1)$ are applied. This

causes the highest ordered part of the sum of the two additional terms involving n to take the form of a product of two series by use of the Christoffel-Darboux formula. This product is summable (C, j), $j>\frac{1}{2}$, to the value zero, so the total order of summability is $k>i-\frac{1}{2}$. (Received November 5, 1934.)

363. Professor A. A. Albert: *Involutorial simple algebras and real Riemann matrices.*

A simple algebra A over F is said to be J-involutorial if there is a self correspondence J of A such that $(a+b)^J=a^J+b^J$, $(ab)^J=b^Ja^J$, $(a^J)^J=a$, $\lambda^J=\lambda$ for every a and b of A and λ of F. The structure problem for J-involutorial simple algebras is closely connected with the study of the multiplication algebras of real Riemann matrices. The author has completed a study of both of these problems and also has shown how closely related they are to Weyl's recent generalization of Riemann matrices. A complete reduction of any impure real Riemann matrix to pure components has also been obtained. (Received November 2, 1934.)

364. Professor C. C. Camp: *A generalization of Chevilliet's formula.*

When the third derivative has the same value at both ends of the interval the error in Simpson's Rule is no longer given by the formula of M. Chevilliet. Consequently when n, the number of strips, is doubled, the second error is not approximately the difference of the computed results divided by 2^4-1 as given by Scarborough (*Numerical Mathematical Analysis*, p. 161). For his case of $f(x)=1/(1+x^2)$ the error varies inversely as n^6. The new error accordingly is nearly the difference divided by 2^6-1. By starting with $n=10$, doubling twice and carrying the computation to 16 decimals, this ratio may be determined empirically to be near a power of 2. Then the last error may be estimated correctly to sixteen decimal places. The ratio rule may be generalized for other calculations in which n is doubled; e.g., to calculate $\sum_1^\infty u_n$ by the approximate formula $\sum_1^{n-1}u_n+\int_n^\infty u_n dn+\frac{1}{2}u_n$ for two values of n. For $u_n=n^{-p}$, $p>1$, the error varies as n^{-p-1}. The ratio rule will apply for any convergent sequence of calculated values where the error varies asymptotically as a power of n and n is sufficiently large. (Received November 2, 1934.)

365. Professor C. C. Camp: *Note on numerical evaluation of double series.*

The Euler-Maclaurin summation formula has been extended, by Dr. Sheppard in 1900 and Irwin in 1923, to two variables to determine cubature formulas. A more complicated two-dimensional form, for which a remainder term was also calculated, was given by Baten in 1932 in the American Journal of Mathematics. The purpose of this paper is to apply the simpler formula to the numerical evaluation of double series of positive terms by the use of integrals and derivatives. If the double series converges one may sum by rows, or columns, using the ordinary sum formula twice. Such series as $\sum\sum(m^2+a^2n^2)^{-2}$ may be summed in this way by taking out a square block of

n^2 terms and applying the formula to the rest; $u(m, n)=(m+n)^{-p}$, $p>2$, may be summed diagonally and thus converted to the single series $\sum n(n+1)^{-p}$. For certain double series lower and upper bounds may be interpreted geometrically in terms of volumes of truncated prisms associated with a surface $z=u(x, y)$. (Received November 2, 1934.)

366. Professor M. G. Gaba: *Finite geometries.* Preliminary report.

The usual approach to non-euclidean geometries has been through the concept of distance, and consequently non-euclidean finite geometries have been ignored. In this paper finite non-euclidean geometries are set up and it is shown that many of the theorems of the classical non-euclidean geometries hold. There are also exhibited finite geometries to fit various coordinate systems which possess peculiarities in their ideal regions. (Received November 2, 1934.)

367. Professor M. A. Basoco: *On certain systems of polynomials.*

Nörlund (Acta Mathematica, 43 (1922), pp. 121–196) has introduced certain systems of polynomials which generalize the classical Bernoulli and Euler polynomials. More recently, Milne-Thomson (Proceedings of the London Mathematical Society, 35 (1933)), has studied these in connection with generalizations of the Hermite polynomials. In the present paper, further properties of these systems are obtained through the application of theorems indicated by Appell in his discussion of a very general class of polynomials (Annales Scientifiques de l'Ecole Normale Supérieure, 2, vol. 9, (1880)). In particular, the Fourier series representation in the interval $0\leqq x\leqq 1$ of this general class of polynomials is obtained and the results are applied to the special instances due to Nörlund and Milne-Thomson. (Received November 2, 1934.)

368. Professor M. A. Basoco: *On a certain identity due to Hermite.*

In a letter to Stieltjes (*Correspondance de Hermite et Stieltjes*, vol. 2, p. 273; Lettre 263) Hermite states without proof an identity involving a certain functional of two arbitrary functions. In the present paper, similar identities involving an arbitrary number of functions are established. These find an immediate application in the theory of the doubly periodic functions of the third kind, the Jacobi theta functions, etc. From these it is possible to obtain a series of identities from which various arithmetical theorems may be deduced. (Received November 2, 1934.)

369. Professor M. A. Basoco: *Arithmetized expansions for certain pseudoperiodic functions.*

In the present paper we are concerned with the derivation of the Fourier expansions for the functions $Z_{\alpha\beta}(x, y)=\theta_1'(0, q)\theta_\alpha(2x+2y, q^2)/\theta_\beta(x, q)$, $T_{\alpha\beta}(x, y; n)=n\theta_1'(0, q^n)\theta_\alpha(x+y, q)/\theta_\beta(nx, q^n)$, $(\alpha, \beta=0, 1, 2, 3)$, where n is a

positive integer >1, and $\theta_\alpha(z, q)$ are the Jacobi elliptic theta functions. As is the case with developments of this sort, they are of interest in connection with arithmetical applications. It is found that the functions $Z_{\alpha\beta}(x, y)$ have an arithmetical structure related to an *indefinite* quadratic form. So far as is known no other expansions of functions of *two* variables have been given with an arithmetical structure of this type. The functions $T_{\alpha\beta}(x, y; n)$ are of interest because they involve irrationalities related to the algebraic number field $K(\sin \pi/n)$, and therefore offer the possibility of deriving (for particular values of n) arithmetical results by equating coefficients of linearly independent irrationals, in identities involving the $T_{\alpha\beta}(x, y; n)$. (Received November 2, 1934.)

370. Professor M. A. Basoco and E. T. Bell: *Further theta expansions useful in arithmetic.*

In a paper with similar title (E. T. Bell, Messenger of Mathematics, vol. 54 (1924), pp. 166–176) several such expansions for functions of one variable have been given, all being deduced from those of the so-called doubly periodic functions of the second kind $\theta_1'(\theta_a(x+y)/\theta_b(x)\theta_c(y))$, $(a, b, c=0, 1, 2, 3)$, where the triple index (abc) has certain sixteen values. These were first obtained by Hermite and Kronecker. For the full and efficient use of the method of paraphrase, it is necessary to have the trigonometric expansions of the remaining forty-eight functions. Following a method suggested elsewhere (M. A. Basoco, American Journal of Mathematics, vol. 54 (1932), pp. 242–252) these expansions in two distinct arithmetical forms have been obtained. In neither set of forty-eight expansions has it been possible to refer to the divisors of a single integer, as is the case for Hermite's sixteen. Instead, the solution in positive integers of equations of the form $n=xy+zw$, $n=xy+2zw$, etc., are required. All sixty-four expansions, when used in paraphrase, have the desirable effect of introducing cross product terms into the quadratic forms appearing in the partitions. (Received November 2, 1934.)

371. Dr. E. W. Anderson: *Statics of special types of homogeneous elastic slabs with variable thickness.*

The derivation and solution of the fourth order partial differential equation for the deflection $z(x, y)$ of the middle surface of a thin rectangular shell of variable thickness $(t(y)=t_1e^{cy}$, $c=(1/L)\log(t_2/t_1))$ for which the upper and lower surfaces are defined by $z=\pm t/2$ are carried out for the following cases: (1) a uniformly loaded, semi-infinite slab $(-\infty \leqq x \leqq \infty, 0 \leqq y \leqq L)$ with simply supported or clamped edges and an initially cylindrical middle surface with large radius of curvature; (2) a simply supported, rectangular plate $(0 \leqq x \leqq a$, $0 \leqq y \leqq L)$ with plane central surface subjected to rectangular, line, or point loads. The solution is effected first by a complex contour integral method and then by a Fourier series method. By setting $t=$constant the results are shown to be in agreement with known solutions of plates of uniform thickness. (Received November 5, 1934.)

372. Mr. J. Biggerstaff: *Solutions of certain ordinary and partial differential equations by means of operators.*

In volume 12 of the "Proceedings of the London Mathematical Society," Glaisher solves certain partial differential equations by using Poisson's integral $e^{a^2} = \int_{-\infty}^{\infty} e^{-u^2+2au} du$. This scheme of using a definite integral to obtain the result of operating on an arbitrary function with an operator which occurs transcendentally is only one among many others which can be used. The writer has had considerable success in solving partial differential equations and systems of partial differential equations which reduce to the result of operating with a transcendental operator. The results are very easily obtained (requiring only a table of definite integrals), and are in the form of definite integrals. They are not necessarily unique. As special cases we obtain the solutions of the differential equations which occur in operational circuit analysis and which have been solved by a much longer process, usually for special cases, in the books on this subject by Bush, Berg, and Cohen. (Received October 17, 1934.)

373. Professor A. F. Moursund: *On the sum of the r th derived series of the conjugate Fourier series.*

When $f(x)$ is Lebesgue integrable on $(-\pi, \pi)$ and of period 2π, there are two equivalent Cauchy integrals either of which may serve as a formula for the sum of the conjugate Fourier series (see G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 211–246). In this note we give a direct proof of the equivalence of two analogous Cauchy integrals which we use as formulas for the sum of the r th derived series of the conjugate Fourier series. We give also a theorem concerning the Bosanquet-Linfoot summability of the r th derived conjugate series. (Received October 29, 1934.)

374. Professor E. T. Bell: *Arithmetical theorems on Lucas functions and Tchebycheff polynomials.*

A simple isomorphism between the Lucas functions, or the Tchebycheff polynomials, and the circular functions, distinct from that of Lucas (American Journal of Mathematics, vol. 1, 1878, p. 189, equation (5)) is first developed. By this means the translation of trigonometric identities into unique correspondents in Lucas functions is immediate. It is then shown that any identity in elliptic, or elliptic theta functions, implies and is implied by an identity in Tchebycheff polynomials (one complex argument) or Lucas functions (two complex arguments). (Received November 1, 1934.)

375. Dr. R. D. James: *Waring's problem for rational numbers.* Preliminary report.

Let $v(k)$ denote the least value of r such that all rational numbers are sums of r rational k-th powers $\geqq 0$. It is known that $v(2) = 4, v(3) = 3$ (see H. W. Richmond, Proceedings of the London Mathematical Society (2), vol. 21 (1922–23), pp. 401–409), but for $k \geqq 4$ nothing seems to be known. In this paper an upper bound for $v(k)$ is obtained. It was pointed out to the author by H. S. Zuckerman that any result concerning the representation of all *sufficiently large* integers as a sum of integral k-th powers implies a similar result for the representation of

all rational numbers as a sum of rational k-th powers. Hence $v(k) \leqq G(k)$, where $G(k)$ denotes the least value of n such that all sufficiently large integers are sums of r integral k-th powers. Using the Hardy-Littlewood result for $G(k)$ (Mathematische Zeitschrift, vol. 23 (1925), pp. 1–37) this gives $v(k) \leqq (k-2)2^{k-2} + k + 5 + \zeta_k$, where $\zeta_k = [\{(k-2)\log 2 - \log k + \log(k-2)\}/\{\log k - \log(k-1)\}]$. When k is odd and $\geqq 5$ the author's result for $G(k)$ (Proceedings of the London Mathematical Society (2), vol. 37 (1934), pp. 359–291) gives $v(k) \leqq (k-3)2^{k-2} + k + 9 + \zeta_k$. A modification of the Gelbeke method (Mathematische Annalen, vol. 105 (1931), pp. 637–652) leads to the inequality $v(k) \leqq (k-2)2^{k-2} + k + 4 + \zeta_k$. This gives, for example, $v(4) \leqq 18$. (Received November 2, 1934.)

376. Dr. C. B. Morrey, Jr.: *On the differentiability of the solutions of a class of regular two dimensional minimum problems.*

Let $f(p, q)$ be a function defined, of class C'', and satisfying $f_{pp}f_{qq} - f_{pq}^2 > 0$ for all real values of p and q, and whose second derivatives satisfy a Hölder condition; let C be a convex curve bounding the region R and let Γ be a curve above C such that any plane through three points of Γ makes an angle $\leqq \tan^{-1}\Delta$ with the (x, y) plane. Radó and Haar have shown the existence and uniqueness of a surface $z = z(x, y)$ bounded by Γ and such that $z(x, y)$ satisfies a Lipschitz condition with the constant Δ and minimizes $\iint_R f(p, q)dxdy$, among all functions satisfying any Lipschitz condition on R and taking on these boundary values. The present paper proves that this solution possesses continuous first partial derivatives which satisfy a Hölder condition on every closed subregion of R. From results due to Lichtenstein and E. Hopf, we may conclude that z_{xx}, z_{xy}, and z_{yy} are continuous and satisfy a Hölder condition, and if $f(p, q)$ is analytic, we may further conclude that $z(x, y)$ is analytic. The solution $z(x, y)$ satisfies the Euler-Lagrange equation $f_{pp}z_{xx} + 2f_{pq}z_{xy} + f_{qq}z_{yy} = 0$ everywhere in R. (Received November 2, 1934.)

377. Dr. Raymond Garver: *Postulates for groups and for commutative groups.*

This paper presents a three-postulate definition of group, and a two-postulate definition of commutative group. (Received November 3, 1934.)

378. Professor E. R. Hedrick: *Generalizations of the Mittag-Leffler theorem and allied theorems.*

The usual theorem regarding the representation of an analytic function that has a finite number of singularities by means of the sum of the Laurent developments is extended in this paper to functions that are not assumed to be analytic, but only to possess continuous second partial derivatives, at points other than the singular points. A similar theorem for non-analytic functions is stated when the singularities are of a much more general type, the only restrictions being that the function behave, near each singular point, like a function that has a singularity at that point only. Finally, the Mittag-Leffler theorem that there exists a function with any preassigned Laurent developments at any preassigned set of singular points that have no finite limiting point is generalized

by replacing the Laurent development at any one singular point by a development whose real part is the real part of some Laurent development, and whose imaginary part is the imaginary part of some (other) Laurent development. (Received November 3, 1934.)

379. Mr. Ivar Highberg: *The existence of rings in vectorial spaces.*

In every vector space real or complex, not necessarily normed, there is shown to exist a symmetric bilinear function, $B(x, y)$, on E^2 to E, thus analogous to a multiplication of elements, and with the associative property $B[x, B(y, z)] = B[B(x, y), z]$. Further properties of this function are considered. (Received November 3, 1934.)

380. Mr. Ivar Highberg: *The existence of a Hermitian inner product in complex vectorial spaces.*

It is proved that in every vector space, closed under multiplication by complex numbers, but not necessarily normed, there exists a function $Q(x, y)$ on E^2 to C, which has the five properties that characterize the Hermitian inner product in Hilbert space. The relation between convergence defined by the norm $[Q(x, x)]^{1/2}$ and an arbitrarily defined norm is considered. (Received November 3, 1934.)

381. Mr. William A. Mersman: *A generalization of the Lebesgue integral.*

The purpose of this paper is to give a definition of an integral of abstract-valued functions of an abstract variable which will be a generalization of similar existing theories. This is attained by letting the range of the variable be a space in which a measure function is assumed to exist, while the values of the function are elements of a vector space. After defining a measurable function as the limit of a sequence of step-functions, we demonstrate the existence of the integral of a measurable, bounded function over a bounded set. By means of Fréchet's (Bulletin de la Société Mathématique de France, vol. 43) extension of the Borel-Lebesgue theorem, continuous functions of a certain type are shown to be integrable. Finally, in case the variable is n-dimensional real or in case the values of the function are real, our definition of an integral is shown to be equivalent to those of Bochner (Fundamenta Mathematicae, Vol. 20, 1933) and Saks (Saks: *Théorie de l'Intégrale*, Warsaw, 1933). (Received November 3, 1934.)

382. Mr. William A. Mersman: *A new type of abstract integral.*

The purpose of the paper is to develop an integral of abstract-valued functions of an abstract variable which will be less artificial in its fundamental definitions than any hitherto considered. This is done by means of a simple generalization of the definition of a measurable function as given by Titchmarsh (Titchmarsh: *The Theory of Functions*, Oxford, 1932). Although our definition is shown to be equivalent to those of Saks (Saks: *Theorie de*

l'Integrale, Warsaw, 1933) and Bochner (Bochner: Fundamenta Mathematicae, vol. 20, 1933) in the special cases which they consider, it permits simpler methods of proof. Some interesting results are obtained regarding functions of two variables, one being measurable and the other in a metric space. For functions of a single variable the integral is shown to be a linear, additive, continuous functional as well as a completely additive function of sets. Finally, the definition is extended so that any bounded, convergent series of integrable functions can be integrated term by term. (Received November 3, 1934.)

383. Professor A. D. Michal: *Abstract euclidean spaces and semi-vector spaces.* Preliminary report.

In an abstract Hilbert space the relation between the norm and the Hermitian form (f, g) is given by $\|f\| = (f, f)^{1/2}$. In this paper postulates are given for normed vector spaces with a continuous Hermitian form in which this relation need not be satisfied. Real as well as complex cases are considered. The second part of the paper deals with modified normed vector spaces for which the postulate $\|\alpha \odot f\| \leq |\alpha| \|f\|$ takes the place of $\|\alpha \odot f\| = |\alpha| \|f\|$. Semi-vectorial rings are also considered. An interesting instance is given by a commutative non-associative vectorial ring of Hermitian matrices. (Received November 3, 1934.)

384. Professor A. D. Michal: *Theorems on integral invariants of associated type.*

The Cartan complete integral invariants are special integral invariants of the associated type. In this paper the following theorem, among others, is proved: a necessary and sufficient condition that I_p^* be an associated integral invariant is that it be in the form $I_p^* = I_p^c + (I_{p-1}, \delta t)$ where I_p^c is a Cartan complete integral invariant and the term in parentheses denotes the symbolic integral product of a $p-1$ dimensional Poincare integral invariant and δt. (Received November 3, 1934.)

385. Professor A. D. Michal: *A theorem on total Fréchet differentials in abstract vector spaces.*

In this paper the author proves the following theorem. If (a) $f(x, y_1, y_2, \cdots, y_n)$ is linear (additive and continuous) in each y_i; (b) the partial Fréchet differential $f_x(x, y_1, y_2, \cdots, y_n; x)$ is continuous in x for x in $\|x - x_0\| < a$ and continuous separately in each y_i, then the total Fréchet differential of $f(x, y_1, \cdots, y_n)$ exists and is continuous jointly in all its variables for x in $\|x - x_0\| < a$ and all δx, y_i, and δy_i. (Received November 3, 1934.)

386. Professor A. D. Michal and Mr. V. Elconin: *Solutions of total differential equations in abstract vector spaces as functions of the initial parameters.*

The authors continue their studies of completely integrable Pfaffian equations $\delta\varphi(x) = F(x, \varphi(x), \delta x)$ in Banach spaces (see this Bulletin, abstract 40-7-228). The solution function $\varphi(x, x_{0i}, \varphi_0)$ taking on the initial value φ_0 for

$x=x_0$ is shown to possess partial Fréchet differentials $\delta_{x_0}\phi(x, x_0, \phi_0; \delta x_0)$ and $\delta_{\phi_0}\varphi(x, x_0, \varphi_0; \delta\varphi_0)$. (Received November 3, 1934.)

387. Mr. Victor Elconin: *Multiple differentials and integrals of abstractly-valued functions.*

Multiple differentials and integrals of functions on an n-space to a Banach space are defined and their relations investigated. Conditions on the integrand are found, necessary and sufficient for the existence of the integral, and sufficient for its continuity and differentiability with respect to the limits of integration and the integrand parameters. The commutativity of multiple integral operators with multiple differential operators that act only on the integrand is established for continuous differentiable integrands. It is shown that a change in the order of repeated multiple integration leaves the integral unaltered if the integrand is continuous over the range of integration. Interesting relations between differentials and integrals of different multiplicities are found, including generalizations of theorems on differentials due to M. Fréchet, A. D. Michal, and M. Kerner. (Received November 3, 1934.)

388. Mr. A. E. Taylor: *A reduced set of postulates for Hilbert space.* Preliminary report.

The usual postulates for abstract Hilbert space contain several redundancies. A reduced set is obtained, and questions of consistency, independence, and categoricalness are examined. (Received November 3, 1934.)

389. Mr. A. E. Taylor: *Integral invariants and integrals of systems of first order differential equations.*

The use of a new type of integral invariant leads to results which are new. Especially interesting is the case of associated integral invariants of nth order, for then we get two multipliers, and hence an integral of the system. From this result, using a theorem of Koenigs, we deduce a second integral. (Received November 3, 1934.)

390. Mr. A. E. Taylor: *Additions to the theory of integral invariants.*

This paper is largely concerned with a new type of integral invariant, which I have called an *associated* invariant. It is an integral invariant, in the Poincaré sense, of an associated differential system, It bears an important relation to Cartan's *complete* integral invariant, which is an attached Poincaré integral invariant of the same associated system. Necessary and sufficient conditions are deduced, and various fundamental relationships are established. In the case of first order *relative* invariants, the theory is developed independently of *absolute* integral invariants, and the use of Stokes's Theorem in hyperspace is avoided. (Received November 3, 1934.)

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

LINCOLN, NEBRASKA, November 30-December 1, 1934.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than November 3. All these will appear in the November issue of this Bulletin. By invitation of the Program Committee, addresses will be delivered by Professor G. A. Bliss on *The problem of Bolza in the calculus of variations*, and by Professor A. J. Kempner on *Complex roots of algebraic equations*.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, December 1, 1934.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than November 3. All these will appear in the November issue of this Bulletin.

PITTSBURGH, PENNSYLVANIA, Annual Meeting, December 27-31, 1934.

This meeting is to be held in conjunction with Section A of the A.A.A.S. and the M.A.A. Professor A. B. Coble will deliver his retiring presidential address. There will be a joint session with the American Physical Society, at which time there will be a symposium on mathematical physics with special reference to quantum mechanics. Professor Albert Einstein will deliver the eleventh Josiah Willard Gibbs Lecture. At a joint session with Section A of the A.A.A.S. and the Mathematical Association, addresses will be delivered by Professor C. N. Moore, as retiring Chairman of Section A, on *Mathematics and Science*, and by Professor Arnold Dresden as retiring President of the Mathematical Association. Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than November 22, 1934. Abstracts received by the Secretaries by November 9 will appear in the November issue of this Bulletin.

NEW YORK CITY, February 23, 1935.

Abstracts will be due on or before January 26, 1935.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-459 Ahnaip St., Menasha, Wis., or 501 West 116th St., New York.

The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1934, pp. 66-67), are payable to the Treasurer of the Society, Professor G. W. Mullins, 501 West 116th St., New York City.

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